The zeta function of $\mathfrak{g}_{1357B}$ counting ideals

1 Presentation

$\mathfrak{g}_{1357B}$ has presentation

$$\left\langle x_1, x_2, x_3, x_4, x_5, x_6, x_7 \;\middle|\; \begin{array}{c} [x_1, x_2] = x_4, [x_1, x_4] = x_5, [x_1, x_5] = x_7, \\ [x_2, x_3] = x_5, [x_3, x_4] = -x_7, [x_3, x_6] = x_7 \end{array} \right\rangle.$$

$\mathfrak{g}_{1357B}$ has nilpotency class 4.

2 The local zeta function

The local zeta function was first calculated by Luke Woodward. It is

$$\begin{aligned} \zeta^{\triangleleft}_{\mathfrak{g}_{1357B},p}(s) = {} & \zeta_p(s)\zeta_p(s-1)\zeta_p(s-2)\zeta_p(s-3)\zeta_p(3s-4)\zeta_p(5s-5)\zeta_p(7s-4) \\ & \times \zeta_p(9s-6)\zeta_p(11s-10)\zeta_p(16s-11)W(p, p^{-s}) \end{aligned}$$

where $W(X, Y)$ is

$$\begin{aligned} & 1 - X^4Y^8 + X^5Y^8 - X^5Y^9 - X^9Y^{11} + X^9Y^{12} - X^{10}Y^{12} - X^{10}Y^{16} \\ & + X^{10}Y^{17} - X^{11}Y^{17} + X^{14}Y^{19} - X^{15}Y^{19} + X^{15}Y^{20} + X^{15}Y^{25} + X^{19}Y^{27} \\ & - X^{19}Y^{28} + X^{21}Y^{28} - X^{25}Y^{36}. \end{aligned}$$

$\zeta^{\triangleleft}_{\mathfrak{g}_{1357B}}(s)$ is uniform.

3 Functional equation

The local zeta function satisfies no functional equation.

4 Abscissa of convergence and order of pole

The abscissa of convergence of $\zeta^{\triangleleft}_{\mathfrak{g}_{1357B}}(s)$ is 4, with a simple pole at $s = 4$.

5 Ghost zeta function

The ghost zeta function is the product over all primes of

$$\begin{aligned} & \zeta_p(s)\zeta_p(s-1)\zeta_p(s-2)\zeta_p(s-3)\zeta_p(3s-4)\zeta_p(5s-5)\zeta_p(7s-4)\zeta_p(9s-6) \\ & \times \zeta_p(11s-10)\zeta_p(16s-11)W_1(p, p^{-s})W_2(p, p^{-s})W_3(p, p^{-s})W_4(p, p^{-s}) \end{aligned}$$

where

$$\begin{aligned}
W_1(X,Y) &= 1 - X^{10}Y^{12},\\
W_2(X,Y) &= -1 - X^5Y^7,\\
W_3(X,Y) &= -1 + X^6Y^9,\\
W_4(X,Y) &= 1 - X^4Y^8.
\end{aligned}$$

The ghost is friendly.

6 Natural boundary

$\zeta^{\triangleleft}_{\mathfrak{g}_{1357B}}(s)$ has a natural boundary at $\Re(s) = 5/6$, and is of type III.

7 Notes

This ideal zeta function is identical to that of $\mathfrak{g}_{1357C}$, though the Lie rings themselves are non-isomorphic.