The zeta function of $\mathfrak{g}_{1357G}$ counting ideals

1 Presentation

$\mathfrak{g}_{1357G}$ has presentation

$$\left\langle x_1, x_2, x_3, x_4, x_5, x_6, x_7 \;\middle|\; \begin{array}{c} [x_1, x_2] = x_3, [x_1, x_4] = x_6, [x_1, x_6] = x_7, \\ [x_2, x_3] = x_5, [x_2, x_5] = x_7 \end{array} \right\rangle.$$

$\mathfrak{g}_{1357G}$ has nilpotency class 4.

2 The local zeta function

The local zeta function was first calculated by Luke Woodward. It is

$$\begin{aligned}
\zeta^{\triangleleft}_{\mathfrak{g}_{1357G},p}(s) = {} & \zeta_p(s)\zeta_p(s-1)\zeta_p(s-2)\zeta_p(3s-4)\zeta_p(4s-3)\zeta_p(5s-5)\zeta_p(5s-6) \\
& \times \zeta_p(6s-6)\zeta_p(7s-4)\zeta_p(7s-7)\zeta_p(8s-5)\zeta_p(9s-6) \\
& \times \zeta_p(10s-9)\zeta_p(11s-8)\zeta_p(12s-10)\zeta_p(12s-11)W(p, p^{-s})
\end{aligned}$$

where $W(X, Y)$ is

$$\begin{aligned}
& 1 + X^{3}Y^{3} - X^{3}Y^{5} - 2X^{6}Y^{7} - 2X^{4}Y^{8} + X^{5}Y^{8} - X^{7}Y^{8} + X^{4}Y^{9} - 2X^{5}Y^{9} \\
& - X^{9}Y^{9} - X^{7}Y^{10} - X^{10}Y^{10} - X^{7}Y^{11} - X^{8}Y^{11} + X^{9}Y^{11} - X^{10}Y^{11} \\
& + 3X^{7}Y^{12} - 3X^{8}Y^{12} - 2X^{9}Y^{12} + X^{10}Y^{12} - X^{7}Y^{13} + 3X^{8}Y^{13} + X^{10}Y^{13} \\
& - 3X^{11}Y^{13} + X^{9}Y^{14} + X^{10}Y^{14} - X^{11}Y^{14} + X^{13}Y^{14} + 5X^{11}Y^{15} - 2X^{12}Y^{15} \\
& - X^{14}Y^{15} + X^{11}Y^{16} + 5X^{12}Y^{16} - 2X^{14}Y^{16} + X^{16}Y^{16} + X^{9}Y^{17} - X^{12}Y^{17} \\
& + X^{13}Y^{17} + 7X^{14}Y^{17} - X^{16}Y^{17} + X^{12}Y^{18} + 2X^{14}Y^{18} + 2X^{15}Y^{18} \\
& + X^{16}Y^{18} + 2X^{12}Y^{19} - 4X^{14}Y^{19} + 6X^{15}Y^{19} + X^{16}Y^{19} + 3X^{17}Y^{19} \\
& - X^{12}Y^{20} + 4X^{13}Y^{20} - 3X^{15}Y^{20} + 2X^{16}Y^{20} + X^{17}Y^{20} + 3X^{18}Y^{20} \\
& - X^{12}Y^{21} - 2X^{13}Y^{21} + 4X^{14}Y^{21} + X^{15}Y^{21} - 2X^{17}Y^{21} + X^{18}Y^{21} \\
& + 2X^{19}Y^{21} + X^{20}Y^{21} - 2X^{14}Y^{22} + X^{16}Y^{22} + X^{20}Y^{22} + X^{21}Y^{22} \\
& - 3X^{15}Y^{23} + X^{16}Y^{23} + 3X^{17}Y^{23} - 2X^{18}Y^{23} - X^{19}Y^{23} - X^{20}Y^{23} \\
& + 2X^{21}Y^{23} - 5X^{16}Y^{24} - 3X^{18}Y^{24} + 6X^{19}Y^{24} - X^{20}Y^{24} - 6X^{21}Y^{24} \\
& + X^{22}Y^{24} + X^{16}Y^{25} - 5X^{17}Y^{25} - X^{18}Y^{25} - 8X^{19}Y^{25} + 6X^{20}Y^{25} \\
& + X^{21}Y^{25} - 2X^{22}Y^{25} + X^{17}Y^{26} - 2X^{18}Y^{26} - 4X^{19}Y^{26} - 6X^{20}Y^{26} \\
& + X^{21}Y^{26} - X^{23}Y^{26} - 2X^{24}Y^{26} + X^{18}Y^{27} - X^{19}Y^{27} - 9X^{20}Y^{27} \\
& - 3X^{21}Y^{27} - 2X^{22}Y^{27} + X^{23}Y^{27} - X^{25}Y^{27} - X^{17}Y^{28} + 2X^{19}Y^{28}
\end{aligned}$$

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\begin{aligned}
&+ X^{20}Y^{28} - 7X^{21}Y^{28} - 8X^{22}Y^{28} - 3X^{23}Y^{28} + 2X^{24}Y^{28} - X^{26}Y^{28}\\
&- X^{27}Y^{28} - X^{19}Y^{29} + 4X^{21}Y^{29} + X^{22}Y^{29} - 11X^{23}Y^{29} - 4X^{24}Y^{29}\\
&- 2X^{25}Y^{29} + X^{26}Y^{29} - 2X^{21}Y^{30} + 4X^{22}Y^{30} - X^{23}Y^{30} - 5X^{24}Y^{30}\\
&- 5X^{25}Y^{30} - 4X^{26}Y^{30} + X^{27}Y^{30} - X^{21}Y^{31} - X^{22}Y^{31} + 8X^{23}Y^{31}\\
&- X^{24}Y^{31} - 9X^{26}Y^{31} - X^{27}Y^{31} + X^{20}Y^{32} + X^{21}Y^{32} - 2X^{22}Y^{32}\\
&- 2X^{23}Y^{32} + 3X^{24}Y^{32} + 5X^{25}Y^{32} + 7X^{26}Y^{32} - 10X^{27}Y^{32} - X^{28}Y^{32}\\
&- 2X^{29}Y^{32} + X^{21}Y^{33} + 2X^{22}Y^{33} - 4X^{25}Y^{33} + 6X^{26}Y^{33} + 8X^{27}Y^{33}\\
&- 5X^{28}Y^{33} - X^{29}Y^{33} - X^{30}Y^{33} + 3X^{23}Y^{34} + X^{24}Y^{34} + 3X^{25}Y^{34}\\
&- X^{26}Y^{34} + X^{27}Y^{34} + 6X^{28}Y^{34} + 2X^{29}Y^{34} - 2X^{30}Y^{34} - X^{31}Y^{34}\\
&+ 5X^{24}Y^{35} + X^{26}Y^{35} + X^{27}Y^{35} + 3X^{28}Y^{35} + 9X^{29}Y^{35} - X^{30}Y^{35}\\
&- X^{32}Y^{35} - 2X^{24}Y^{36} + 5X^{25}Y^{36} + 4X^{26}Y^{36} + 10X^{27}Y^{36} - 8X^{28}Y^{36}\\
&- 5X^{29}Y^{36} + 12X^{30}Y^{36} + X^{31}Y^{36} + 4X^{32}Y^{36} - X^{33}Y^{36} - X^{25}Y^{37}\\
&+ 2X^{26}Y^{37} + 15X^{28}Y^{37} - X^{29}Y^{37} - 2X^{30}Y^{37} + 4X^{31}Y^{37} + X^{32}Y^{37}\\
&+ 3X^{33}Y^{37} - X^{26}Y^{38} - X^{27}Y^{38} + 2X^{28}Y^{38} + 13X^{29}Y^{38} + 2X^{30}Y^{38}\\
&+ X^{31}Y^{38} - X^{32}Y^{38} + 4X^{33}Y^{38} + X^{34}Y^{38} + X^{35}Y^{38} - 2X^{27}Y^{39} - X^{28}Y^{39}\\
&+ 2X^{29}Y^{39} + 9X^{30}Y^{39} + 5X^{31}Y^{39} - X^{33}Y^{39} + 3X^{34}Y^{39} + X^{36}Y^{39}\\
&+ X^{27}Y^{40} - X^{28}Y^{40} - 6X^{29}Y^{40} - 5X^{30}Y^{40} + 13X^{31}Y^{40} + 12X^{32}Y^{40}\\
&- 3X^{33}Y^{40} - X^{34}Y^{40} + X^{35}Y^{40} + X^{36}Y^{40} + X^{37}Y^{40} + X^{29}Y^{41} - 2X^{30}Y^{41}\\
&- 10X^{31}Y^{41} + 8X^{33}Y^{41} + 9X^{34}Y^{41} - 2X^{35}Y^{41} - X^{36}Y^{41} + X^{30}Y^{42}\\
&- 5X^{31}Y^{42} - 4X^{32}Y^{42} - X^{33}Y^{42} + 3X^{34}Y^{42} + 7X^{35}Y^{42} - 2X^{36}Y^{42}\\
&+ X^{37}Y^{42} - X^{29}Y^{43} + 2X^{31}Y^{43} - 5X^{32}Y^{43} - 6X^{33}Y^{43} - 12X^{34}Y^{43}\\
&+ 6X^{35}Y^{43} + 6X^{36}Y^{43} - X^{39}Y^{43} - X^{30}Y^{44} - X^{31}Y^{44} + X^{32}Y^{44}\\
&+ 2X^{33}Y^{44} - 6X^{34}Y^{44} - 18X^{35}Y^{44} + 4X^{36}Y^{44} + 2X^{37}Y^{44} + 3X^{38}Y^{44}\\
&- X^{31}Y^{45} - 3X^{32}Y^{45} - 3X^{33}Y^{45} + 6X^{34}Y^{45} + X^{35}Y^{45} - 15X^{36}Y^{45}\\
&- 8X^{37}Y^{45} - X^{38}Y^{45} + 5X^{39}Y^{45} - X^{40}Y^{45} - 2X^{33}Y^{46} - 4X^{34}Y^{46}\\
&- X^{36}Y^{46} - 6X^{37}Y^{46} - 6X^{38}Y^{46} - 2X^{39}Y^{46} + X^{40}Y^{46} - 3X^{34}Y^{47}\\
&- 7X^{35}Y^{47} + 4X^{36}Y^{47} + 3X^{37}Y^{47} - 12X^{38}Y^{47} - 5X^{39}Y^{47} - 5X^{40}Y^{47}\\
&+ 2X^{41}Y^{47} + 2X^{35}Y^{48} - 12X^{36}Y^{48} + 9X^{38}Y^{48} - 6X^{39}Y^{48} - 10X^{41}Y^{48}\\
&+ X^{42}Y^{48} + X^{35}Y^{49} + 4X^{36}Y^{49} - 10X^{37}Y^{49} - 5X^{38}Y^{49} + X^{39}Y^{49}\\
&+ X^{41}Y^{49} - 7X^{42}Y^{49} - X^{43}Y^{49} + X^{36}Y^{50} + 4X^{37}Y^{50} - 4X^{38}Y^{50}\\
&- 5X^{39}Y^{50} - 5X^{40}Y^{50} + 6X^{41}Y^{50} - X^{42}Y^{50} - 2X^{43}Y^{50} - 2X^{44}Y^{50}\\
&+ X^{37}Y^{51} + 6X^{38}Y^{51} - 2X^{39}Y^{51} - 6X^{40}Y^{51} - 3X^{41}Y^{51} + 5X^{42}Y^{51}\\
&- X^{44}Y^{51} - 2X^{45}Y^{51} + X^{37}Y^{52} + 9X^{39}Y^{52} + 4X^{40}Y^{52} - 7X^{41}Y^{52}\\
&- 7X^{42}Y^{52} + X^{43}Y^{52} + 5X^{44}Y^{52} - X^{45}Y^{52} - X^{46}Y^{52} - 2X^{39}Y^{53}
\end{aligned}
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\begin{aligned}
&+5X^{40}Y^{53}+8X^{41}Y^{53}+4X^{42}Y^{53}-8X^{43}Y^{53}-2X^{44}Y^{53}+3X^{45}Y^{53}\\
&+X^{40}Y^{54}+3X^{41}Y^{54}+6X^{42}Y^{54}+7X^{43}Y^{54}-5X^{44}Y^{54}-X^{45}Y^{54}\\
&+X^{46}Y^{54}+2X^{47}Y^{54}-X^{48}Y^{54}+X^{39}Y^{55}+X^{40}Y^{55}-X^{41}Y^{55}+X^{42}Y^{55}\\
&+10X^{43}Y^{55}+5X^{44}Y^{55}-X^{45}Y^{55}-2X^{46}Y^{55}-2X^{47}Y^{55}+3X^{48}Y^{55}\\
&+X^{41}Y^{56}-2X^{42}Y^{56}-X^{43}Y^{56}+8X^{44}Y^{56}+7X^{45}Y^{56}+5X^{46}Y^{56}\\
&-4X^{47}Y^{56}-X^{48}Y^{56}+X^{49}Y^{56}+3X^{42}Y^{57}-4X^{44}Y^{57}+2X^{45}Y^{57}\\
&+9X^{46}Y^{57}+7X^{47}Y^{57}-2X^{48}Y^{57}+3X^{44}Y^{58}-2X^{46}Y^{58}+4X^{47}Y^{58}\\
&+5X^{48}Y^{58}+X^{44}Y^{59}+X^{45}Y^{59}-3X^{46}Y^{59}+2X^{48}Y^{59}+7X^{49}Y^{59}\\
&-X^{44}Y^{60}+X^{45}Y^{60}+2X^{46}Y^{60}-X^{48}Y^{60}-6X^{49}Y^{60}+7X^{50}Y^{60}\\
&+X^{51}Y^{60}-X^{45}Y^{61}+3X^{48}Y^{61}-X^{49}Y^{61}-5X^{50}Y^{61}+4X^{51}Y^{61}\\
&+X^{52}Y^{61}-2X^{46}Y^{62}-X^{47}Y^{62}-3X^{48}Y^{62}+4X^{49}Y^{62}-4X^{51}Y^{62}\\
&-X^{52}Y^{62}+2X^{53}Y^{62}-2X^{47}Y^{63}+X^{48}Y^{63}-2X^{50}Y^{63}+2X^{51}Y^{63}\\
&-3X^{52}Y^{63}-2X^{53}Y^{63}+2X^{54}Y^{63}+X^{47}Y^{64}-3X^{48}Y^{64}-5X^{49}Y^{64}\\
&-X^{50}Y^{64}+X^{51}Y^{64}+3X^{52}Y^{64}-2X^{53}Y^{64}-2X^{54}Y^{64}-6X^{51}Y^{65}\\
&+X^{53}Y^{65}-X^{54}Y^{65}-X^{51}Y^{66}-6X^{52}Y^{66}-X^{56}Y^{66}-X^{50}Y^{67}-2X^{53}Y^{67}\\
&-X^{54}Y^{67}-X^{55}Y^{67}+X^{56}Y^{67}-X^{57}Y^{67}+2X^{52}Y^{68}-4X^{54}Y^{68}\\
&-3X^{55}Y^{68}+X^{56}Y^{68}-X^{53}Y^{69}+3X^{54}Y^{69}-3X^{56}Y^{69}-X^{57}Y^{69}\\
&-X^{58}Y^{70}+2X^{55}Y^{71}+X^{57}Y^{71}-X^{59}Y^{71}+2X^{58}Y^{72}-X^{59}Y^{72}+X^{56}Y^{73}\\
&-X^{57}Y^{73}+2X^{59}Y^{73}+2X^{57}Y^{74}+X^{60}Y^{74}-X^{60}Y^{75}+2X^{61}Y^{75}\\
&+X^{59}Y^{76}+X^{60}Y^{76}-X^{61}Y^{76}+X^{62}Y^{76}+X^{60}Y^{77}+X^{63}Y^{78}-X^{63}Y^{81}\\
&-X^{66}Y^{83}.
\end{aligned}
$$

$\zeta^{\triangleleft}_{\mathfrak{g}_{1357G}}(s)$ is uniform.

3 Functional equation

The local zeta function satisfies no functional equation.

4 Abscissa of convergence and order of pole

The abscissa of convergence of $\zeta^{\triangleleft}_{\mathfrak{g}_{1357G}}(s)$ is 3, with a simple pole at $s = 3$.

5 Ghost zeta function

The ghost zeta function is the product over all primes of

$$\begin{aligned}
&\zeta_p(s)\zeta_p(s-1)\zeta_p(s-2)\zeta_p(3s-4)\zeta_p(4s-3)\zeta_p(5s-5)\zeta_p(5s-6)\zeta_p(6s-6)\\
&\times \zeta_p(7s-4)\zeta_p(7s-7)\zeta_p(8s-5)\zeta_p(9s-6)\zeta_p(10s-9)\zeta_p(11s-8)\zeta_p(12s-10)\\
&\times \zeta_p(12s-11)W_1(p,p^{-s})W_2(p,p^{-s})W_3(p,p^{-s})W_4(p,p^{-s})W_5(p,p^{-s})\\
&\times W_6(p,p^{-s})
\end{aligned}$$

where

$$\begin{aligned}
W_1(X,Y) &= 1 + X^3Y^3 - X^9Y^9 - X^{10}Y^{10} + X^{16}Y^{16},\\
W_2(X,Y) &= 1 - X^{11}Y^{12},\\
W_3(X,Y) &= -1 + X^{10}Y^{12},\\
W_4(X,Y) &= 1 - X^{11}Y^{14},\\
W_5(X,Y) &= -1 - X^9Y^{13},\\
W_6(X,Y) &= -1 - X^9Y^{16}.
\end{aligned}$$

The ghost is unfriendly.

6 Natural boundary

$\zeta^{\triangleleft}_{\mathfrak{g}_{1357G}}(s)$ has a natural boundary at $\Re(s) = 1$, and is of type I.

7 Notes

This ideal zeta function is identical to that of $\mathfrak{g}_{1357H}$, though the Lie rings themselves are non-isomorphic.