The zeta function of $\mathfrak{g}_{137D}$ counting ideals

1 Presentation

$\mathfrak{g}_{137D}$ has presentation

$$\left\langle x_1, x_2, x_3, x_4, x_5, x_6, x_7 \;\middle|\; \begin{array}{l} [x_1, x_2] = x_5, [x_1, x_4] = x_6, [x_1, x_6] = x_7, \\ [x_2, x_3] = x_6, [x_2, x_4] = x_7, [x_3, x_5] = -x_7 \end{array} \right\rangle.$$

$\mathfrak{g}_{137D}$ has nilpotency class 3.

2 The local zeta function

The local zeta function was first calculated by Luke Woodward. It is

$$\begin{aligned}
\zeta^{\triangleleft}_{\mathfrak{g}_{137D},p}(s) = {} & \zeta_p(s)\zeta_p(s-1)\zeta_p(s-2)\zeta_p(s-3)\zeta_p(3s-4)\zeta_p(5s-5)\zeta_p(6s-9) \\
& \times \zeta_p(7s-4)\zeta_p(9s-6)\zeta_p(11s-10)\zeta_p(12s-10)\zeta_p(16s-11) \\
& \times W(p, p^{-s})
\end{aligned}$$

where $W(X, Y)$ is

$$\begin{aligned}
& 1 - X^4Y^8 + X^5Y^8 - X^9Y^8 - X^5Y^9 - X^9Y^{11} - X^{10}Y^{12} + X^9Y^{13} - X^{10}Y^{13} \\
& + X^{13}Y^{15} - X^{14}Y^{15} - X^{10}Y^{16} + X^{14}Y^{16} - X^{15}Y^{16} + X^{10}Y^{17} - X^{11}Y^{17} \\
& + X^{15}Y^{17} + X^{14}Y^{19} - X^{15}Y^{19} + X^{19}Y^{19} + X^{15}Y^{20} + X^{19}Y^{20} + X^{14}Y^{21} \\
& + X^{15}Y^{21} - X^{16}Y^{21} - X^{15}Y^{22} + X^{16}Y^{22} + X^{18}Y^{23} + X^{19}Y^{23} - X^{20}Y^{23} \\
& - X^{18}Y^{24} - X^{19}Y^{24} + 3X^{20}Y^{24} + X^{15}Y^{25} - X^{23}Y^{26} + X^{24}Y^{26} + X^{19}Y^{27} \\
& - X^{19}Y^{28} + X^{20}Y^{28} + X^{21}Y^{28} - X^{23}Y^{28} - X^{24}Y^{28} + X^{25}Y^{28} - X^{25}Y^{29} \\
& - X^{20}Y^{30} + X^{21}Y^{30} - X^{29}Y^{31} - 3X^{24}Y^{32} + X^{25}Y^{32} + X^{26}Y^{32} + X^{24}Y^{33} \\
& - X^{25}Y^{33} - X^{26}Y^{33} - X^{28}Y^{34} + X^{29}Y^{34} + X^{28}Y^{35} - X^{29}Y^{35} - X^{30}Y^{35} \\
& - X^{25}Y^{36} - X^{29}Y^{36} - X^{25}Y^{37} + X^{29}Y^{37} - X^{30}Y^{37} - X^{29}Y^{39} + X^{33}Y^{39} \\
& - X^{34}Y^{39} + X^{29}Y^{40} - X^{30}Y^{40} + X^{34}Y^{40} + X^{30}Y^{41} - X^{31}Y^{41} + X^{34}Y^{43} \\
& - X^{35}Y^{43} + X^{34}Y^{44} + X^{35}Y^{45} + X^{39}Y^{47} + X^{35}Y^{48} - X^{39}Y^{48} + X^{40}Y^{48} \\
& - X^{44}Y^{56}.
\end{aligned}$$

$\zeta^{\triangleleft}_{\mathfrak{g}_{137D}}(s)$ is uniform.

3 Functional equation

The local zeta function satisfies the functional equation

$$\zeta_{\mathfrak{g}_{137D},p}^{\triangleleft}(s)\big|_{p \to p^{-1}} = -p^{21-17s}\zeta_{\mathfrak{g}_{137D},p}^{\triangleleft}(s).$$

4 Abscissa of convergence and order of pole

The abscissa of convergence of $\zeta_{\mathfrak{g}_{137D}}^{\triangleleft}(s)$ is 4, with a simple pole at $s = 4$.

5 Ghost zeta function

The ghost zeta function is the product over all primes of

$$\zeta_p(s)\zeta_p(s-1)\zeta_p(s-2)\zeta_p(s-3)\zeta_p(3s-4)\zeta_p(5s-5)\zeta_p(6s-9)\zeta_p(7s-4)$$
$$\times \zeta_p(9s-6)\zeta_p(11s-10)\zeta_p(12s-10)\zeta_p(16s-11)W_1(p,p^{-s})W_2(p,p^{-s})$$
$$\times W_3(p,p^{-s})W_4(p,p^{-s})W_5(p,p^{-s})$$

where

$$W_1(X,Y) = 1 - X^9Y^8,$$
$$W_2(X,Y) = -1 + X^{10}Y^{11},$$
$$W_3(X,Y) = 1 - X^{10}Y^{12},$$
$$W_4(X,Y) = -1 + X^{11}Y^{17},$$
$$W_5(X,Y) = 1 - X^4Y^8.$$

The ghost is friendly.

6 Natural boundary

$\zeta_{\mathfrak{g}_{137D}}^{\triangleleft}(s)$ has a natural boundary at $\Re(s) = 9/8$, and is of type II.

7 Notes

This ideal zeta function is identical to that of $\mathfrak{g}_{137C}$, though the Lie rings themselves are non-isomorphic.