The zeta function of $\mathfrak{g}_{1457A}$ counting ideals

1 Presentation

$\mathfrak{g}_{1457A}$ has presentation

$$\left\langle x_1, x_2, x_3, x_4, x_5, x_6, x_7 \;\middle|\; \begin{array}{c} [x_1, x_2] = x_5, [x_1, x_5] = x_6, \\ [x_1, x_6] = x_7, [x_3, x_4] = x_7 \end{array} \right\rangle.$$

$\mathfrak{g}_{1457A}$ has nilpotency class 4.

2 The local zeta function

The local zeta function was first calculated by Luke Woodward. It is

$$\begin{aligned} \zeta^{\triangleleft}_{\mathfrak{g}_{1457A},p}(s) &= \zeta_p(s)\zeta_p(s-1)\zeta_p(s-2)\zeta_p(s-3)\zeta_p(3s-4)\zeta_p(4s-4)\zeta_p(5s-5) \\ &\times \zeta_p(7s-4)\zeta_p(9s-6)\zeta_p(10s-9)\zeta_p(11s-10)\zeta_p(12s-10) \\ &\times \zeta_p(15s-10)\zeta_p(16s-11)W(p,p^{-s}) \end{aligned}$$

where $W(X,Y)$ is

$$\begin{aligned} &1 - X^{4}Y^{5} - X^{4}Y^{8} + X^{5}Y^{8} - X^{5}Y^{9} - X^{8}Y^{10} + X^{8}Y^{11} - 2X^{9}Y^{11} + X^{8}Y^{12} \\ &- X^{9}Y^{12} - X^{10}Y^{12} + 2X^{9}Y^{13} - 2X^{10}Y^{13} + X^{10}Y^{14} - X^{9}Y^{15} + 2X^{13}Y^{15} \\ &- X^{14}Y^{15} + X^{9}Y^{16} - 2X^{10}Y^{16} - X^{13}Y^{16} + 2X^{14}Y^{16} + X^{10}Y^{17} - X^{11}Y^{17} \\ &+ X^{14}Y^{17} + 2X^{13}Y^{18} - 2X^{14}Y^{18} + 3X^{14}Y^{19} - 2X^{15}Y^{19} - X^{13}Y^{20} \\ &+ 3X^{14}Y^{20} - X^{14}Y^{21} + 4X^{15}Y^{21} - X^{16}Y^{21} + X^{18}Y^{21} - X^{15}Y^{22} + X^{16}Y^{22} \\ &- X^{17}Y^{22} + X^{18}Y^{22} + X^{19}Y^{22} + X^{14}Y^{23} - X^{15}Y^{23} - 3X^{18}Y^{23} + 4X^{19}Y^{23} \\ &+ 2X^{15}Y^{24} - X^{16}Y^{24} - X^{18}Y^{24} - 2X^{19}Y^{24} + 2X^{20}Y^{24} + X^{16}Y^{25} \\ &+ X^{18}Y^{25} - X^{19}Y^{25} - X^{18}Y^{26} + 4X^{19}Y^{26} - 2X^{20}Y^{26} - X^{23}Y^{26} - X^{18}Y^{27} \\ &- X^{19}Y^{27} + 4X^{20}Y^{27} - X^{23}Y^{27} - 3X^{19}Y^{28} + 3X^{20}Y^{28} + X^{21}Y^{28} \\ &- X^{24}Y^{28} - 3X^{20}Y^{29} + 2X^{21}Y^{29} - X^{23}Y^{29} + X^{24}Y^{29} - X^{21}Y^{30} \\ &- 3X^{23}Y^{30} + X^{24}Y^{30} + X^{20}Y^{31} - 5X^{24}Y^{31} + 2X^{25}Y^{31} - X^{20}Y^{32} \\ &+ X^{21}Y^{32} + X^{23}Y^{32} - X^{24}Y^{32} - 3X^{25}Y^{32} - X^{23}Y^{33} + X^{24}Y^{33} - X^{25}Y^{33} \\ &- X^{28}Y^{33} - 3X^{24}Y^{34} + 2X^{25}Y^{34} + X^{27}Y^{34} - X^{29}Y^{34} - X^{24}Y^{35} \\ &- 2X^{25}Y^{35} + X^{26}Y^{35} + X^{27}Y^{35} + X^{28}Y^{35} - X^{29}Y^{35} - 3X^{25}Y^{36} - X^{26}Y^{37} \\ &- X^{28}Y^{37} + X^{27}Y^{38} + X^{28}Y^{38} - 3X^{29}Y^{38} - X^{30}Y^{38} + X^{33}Y^{38} + 3X^{28}Y^{39} \\ &- 3X^{30}Y^{39} - X^{25}Y^{40} + X^{28}Y^{40} + 3X^{29}Y^{40} - X^{30}Y^{40} - X^{31}Y^{40} + X^{30}Y^{41} \end{aligned}$$

$$
\begin{aligned}
&+ X^{32}Y^{41} + 3X^{33}Y^{42} + X^{29}Y^{43} - X^{30}Y^{43} - X^{31}Y^{43} - X^{32}Y^{43} + 2X^{33}Y^{43} \\
&+ X^{34}Y^{43} + X^{29}Y^{44} - X^{31}Y^{44} - 2X^{33}Y^{44} + 3X^{34}Y^{44} + X^{30}Y^{45} + X^{33}Y^{45} \\
&- X^{34}Y^{45} + X^{35}Y^{45} + 3X^{33}Y^{46} + X^{34}Y^{46} - X^{35}Y^{46} - X^{37}Y^{46} + X^{38}Y^{46} \\
&- 2X^{33}Y^{47} + 5X^{34}Y^{47} - X^{38}Y^{47} - X^{34}Y^{48} + 3X^{35}Y^{48} + X^{37}Y^{48} \\
&- X^{34}Y^{49} + X^{35}Y^{49} - 2X^{37}Y^{49} + 3X^{38}Y^{49} + X^{34}Y^{50} - X^{37}Y^{50} \\
&- 3X^{38}Y^{50} + 3X^{39}Y^{50} + X^{35}Y^{51} - 4X^{38}Y^{51} + X^{39}Y^{51} + X^{40}Y^{51} \\
&+ X^{35}Y^{52} + 2X^{38}Y^{52} - 4X^{39}Y^{52} + X^{40}Y^{52} + X^{39}Y^{53} - X^{40}Y^{53} - X^{42}Y^{53} \\
&- 2X^{38}Y^{54} + 2X^{39}Y^{54} + X^{40}Y^{54} + X^{42}Y^{54} - 2X^{43}Y^{54} - 4X^{39}Y^{55} \\
&+ 3X^{40}Y^{55} + X^{43}Y^{55} - X^{44}Y^{55} - X^{39}Y^{56} - X^{40}Y^{56} + X^{41}Y^{56} - X^{42}Y^{56} \\
&+ X^{43}Y^{56} - X^{40}Y^{57} + X^{42}Y^{57} - 4X^{43}Y^{57} + X^{44}Y^{57} - 3X^{44}Y^{58} + X^{45}Y^{58} \\
&+ 2X^{43}Y^{59} - 3X^{44}Y^{59} + 2X^{44}Y^{60} - 2X^{45}Y^{60} - X^{44}Y^{61} + X^{47}Y^{61} \\
&- X^{48}Y^{61} - 2X^{44}Y^{62} + X^{45}Y^{62} + 2X^{48}Y^{62} - X^{49}Y^{62} + X^{44}Y^{63} \\
&- 2X^{45}Y^{63} + X^{49}Y^{63} - X^{48}Y^{64} + 2X^{48}Y^{65} - 2X^{49}Y^{65} + X^{48}Y^{66} \\
&+ X^{49}Y^{66} - X^{50}Y^{66} + 2X^{49}Y^{67} - X^{50}Y^{67} + X^{50}Y^{68} + X^{53}Y^{69} - X^{53}Y^{70} \\
&+ X^{54}Y^{70} + X^{54}Y^{73} - X^{58}Y^{78}.
\end{aligned}
$$

$\zeta^{\triangleleft}_{\mathfrak{g}_{1457A}}(s)$ is uniform.

3 Functional equation

The local zeta function satisfies the functional equation

$$\zeta^{\triangleleft}_{\mathfrak{g}_{1457A},p}(s)\Big|_{p\to p^{-1}} = -p^{21-18s}\zeta^{\triangleleft}_{\mathfrak{g}_{1457A},p}(s).$$

4 Abscissa of convergence and order of pole

The abscissa of convergence of $\zeta^{\triangleleft}_{\mathfrak{g}_{1457A}}(s)$ is 4, with a simple pole at $s = 4$.

5 Ghost zeta function

The ghost zeta function is the product over all primes of

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\begin{aligned}
&\zeta_p(s)\zeta_p(s-1)\zeta_p(s-2)\zeta_p(s-3)\zeta_p(3s-4)\zeta_p(4s-4)\zeta_p(5s-5)\zeta_p(7s-4) \\
&\times \zeta_p(9s-6)\zeta_p(10s-9)\zeta_p(11s-10)\zeta_p(12s-10)\zeta_p(15s-10)\zeta_p(16s-11) \\
&\times W_1(p,p^{-s})W_2(p,p^{-s})W_3(p,p^{-s})W_4(p,p^{-s})W_5(p,p^{-s})
\end{aligned}
$$

where

$$
\begin{aligned}
W_1(X,Y) &= 1 - X^{14}Y^{15},\\
W_2(X,Y) &= -1 + X^{19}Y^{23},\\
W_3(X,Y) &= 1 - X^{16}Y^{24},\\
W_4(X,Y) &= -1 + X^{5}Y^{8},\\
W_5(X,Y) &= 1 - X^{4}Y^{8}.
\end{aligned}
$$

The ghost is friendly.

6 Natural boundary

$\zeta^{\triangleleft}_{\mathfrak{g}_{1457A}}(s)$ has a natural boundary at $\Re(s) = 14/15$, and is of type III.

7 Notes

The Lie ring $\mathfrak{g}_{1457A}$ is isomorphic to the central amalgamation of the maximal class Lie ring M_4 with the Heisenberg Lie ring H.