The zeta function of $\mathfrak{g}_{257B}$ counting ideals

1 Presentation

$\mathfrak{g}_{257B}$ has presentation

$$\langle x_1, x_2, x_3, x_4, x_5, x_6, x_7 \mid [x_1, x_2] = x_3, [x_1, x_3] = x_6, [x_1, x_4] = x_7, [x_2, x_5] = x_7 \rangle.$$ $\mathfrak{g}_{257B}$ has nilpotency class 3.

2 The local zeta function

The local zeta function was first calculated by Luke Woodward. It is

$$\zeta_{\mathfrak{g}_{257B}, \mathfrak{g}_{257B}}(s) = \zeta_p(s)\zeta_p(s-1)\zeta_p(s-2)\zeta_p(s-3)\zeta_p(3s-4)\zeta_p(4s-4)\zeta_p(5s-6) \times \zeta_p(6s-9)\zeta_p(7s-9)\zeta_p(8s-10)\zeta_p(12s-15)W(p, p^{-s})$$

where $W(X,Y)$ is

$$1 - X^4Y^5 + X^6Y^5 - 2X^9Y^8 - X^9Y^9 - X^{13}Y^{10} + X^{13}Y^{11} - X^{14}Y^{11} + 2X^{14}Y^{12} - 2X^{14}Y^{12} + X^{14}Y^{12} - X^{15}Y^{13} + 2X^{18}Y^{15} - X^{19}Y^{15} + X^{18}Y^{16} + 2X^{19}Y^{17} - X^{20}Y^{17} + X^{23}Y^{18} - X^{22}Y^{19} + X^{23}Y^{19} - X^{23}Y^{20} + 2X^{24}Y^{20} + X^{24}Y^{21} + X^{28}Y^{22} - X^{27}Y^{23} - X^{28}Y^{23} + X^{29}Y^{23} - 2X^{28}Y^{24} + X^{29}Y^{24} - X^{33}Y^{27} - X^{33}Y^{28} - X^{33}Y^{29} - X^{38}Y^{30} + X^{37}Y^{32} + X^{42}Y^{35}.$$ $\zeta_{\mathfrak{g}_{257B}}(s)$ is uniform.

3 Functional equation

The local zeta function satisfies no functional equation.

4 Abscissa of convergence and order of pole

The abscissa of convergence of $\zeta_{\mathfrak{g}_{257B}}(s)$ is 4, with a simple pole at $s = 4$. 
5 Ghost zeta function

The ghost zeta function is the product over all primes of

\[
\zeta_p(s)\zeta_p(s-1)\zeta_p(s-2)\zeta_p(s-3)\zeta_p(3s-4)\zeta_p(4s-4)\zeta_p(5s-6)\zeta_p(6s-9)
\times \zeta_p(7s-9)\zeta_p(8s-10)\zeta_p(12s-15)W_1(p,p^{-s})W_2(p,p^{-s})W_3(p,p^{-s})
\]

where

\[
W_1(X,Y) = 1 - X^{13}Y^{10}, \\
W_2(X,Y) = -1 + X^{10}Y^8 + X^{15}Y^{12} - X^{25}Y^{20}, \\
W_3(X,Y) = -1 + X^4Y^5.
\]

The ghost is friendly.

6 Natural boundary

\[\zeta_{\mathbb{P}_{257n}}(s)\] has a natural boundary at \(\Re(s) = 13/10\), and is of type III.