The zeta function of $\mathfrak{g}_{257B}$ counting ideals

1 Presentation

$\mathfrak{g}_{257B}$ has presentation

$$\left\langle x_1, x_2, x_3, x_4, x_5, x_6, x_7 \;\middle|\; \begin{array}{c} [x_1, x_2] = x_3, [x_1, x_3] = x_6, \\ [x_1, x_4] = x_7, [x_2, x_5] = x_7 \end{array} \right\rangle.$$

$\mathfrak{g}_{257B}$ has nilpotency class 3.

2 The local zeta function

The local zeta function was first calculated by Luke Woodward. It is

$$\begin{aligned}\zeta^{\triangleleft}_{\mathfrak{g}_{257B},p}(s) = \zeta_p(s)\zeta_p(s-1)\zeta_p(s-2)\zeta_p(s-3)\zeta_p(3s-4)\zeta_p(4s-4)\zeta_p(5s-6) \\ \times \zeta_p(6s-9)\zeta_p(7s-9)\zeta_p(8s-10)\zeta_p(12s-15)W(p,p^{-s})\end{aligned}$$

where $W(X,Y)$ is

$$\begin{aligned}&1 - X^4Y^5 + X^5Y^5 - 2X^9Y^8 - X^9Y^9 - X^{13}Y^{10} + X^{13}Y^{11} - X^{14}Y^{11} \\ &+ 2X^{13}Y^{12} - 2X^{14}Y^{12} + X^{14}Y^{13} - X^{15}Y^{13} + 2X^{18}Y^{15} - X^{19}Y^{15} \\ &+ X^{18}Y^{16} + 2X^{19}Y^{17} - X^{20}Y^{17} + X^{23}Y^{18} - X^{22}Y^{19} + X^{23}Y^{19} - X^{23}Y^{20} \\ &+ 2X^{24}Y^{20} + X^{24}Y^{21} + X^{28}Y^{22} - X^{27}Y^{23} - X^{28}Y^{23} + X^{29}Y^{23} - 2X^{28}Y^{24} \\ &+ X^{29}Y^{24} - X^{33}Y^{27} - X^{33}Y^{28} - X^{33}Y^{29} - X^{38}Y^{30} + X^{37}Y^{32} + X^{42}Y^{35}.\end{aligned}$$

$\zeta^{\triangleleft}_{\mathfrak{g}_{257B}}(s)$ is uniform.

3 Functional equation

The local zeta function satisfies no functional equation.

4 Abscissa of convergence and order of pole

The abscissa of convergence of $\zeta^{\triangleleft}_{\mathfrak{g}_{257B}}(s)$ is 4, with a simple pole at $s = 4$.

5 Ghost zeta function

The ghost zeta function is the product over all primes of

$$\zeta_p(s)\zeta_p(s-1)\zeta_p(s-2)\zeta_p(s-3)\zeta_p(3s-4)\zeta_p(4s-4)\zeta_p(5s-6)\zeta_p(6s-9)$$
$$\times\,\zeta_p(7s-9)\zeta_p(8s-10)\zeta_p(12s-15)W_1(p,p^{-s})W_2(p,p^{-s})W_3(p,p^{-s})$$

where

$$W_1(X,Y) = 1 - X^{13}Y^{10},$$
$$W_2(X,Y) = -1 + X^{10}Y^8 + X^{15}Y^{12} - X^{25}Y^{20},$$
$$W_3(X,Y) = -1 + X^4Y^5.$$

The ghost is friendly.

6 Natural boundary

$\zeta^{\triangleleft}_{\mathfrak{g}_{257B}}(s)$ has a natural boundary at $\Re(s) = 13/10$, and is of type III.