The zeta function of $\mathfrak{g}_{37D}$ counting ideals

1 Presentation

$\mathfrak{g}_{37D}$ has presentation

$$\left\langle x_1, x_2, x_3, x_4, x_5, x_6, x_7 \;\middle|\; \begin{array}{c} [x_1, x_2] = x_5, [x_1, x_3] = x_7, \\ [x_2, x_4] = x_7, [x_3, x_4] = x_6 \end{array} \right\rangle.$$

$\mathfrak{g}_{37D}$ has nilpotency class 2.

2 The local zeta function

The local zeta function was first calculated by Luke Woodward. It is

$$\begin{aligned} \zeta^{\triangleleft}_{\mathfrak{g}_{37D},p}(s) = {} & \zeta_p(s)\zeta_p(s-1)\zeta_p(s-2)\zeta_p(s-3)\zeta_p(3s-5)\zeta_p(5s-6)\zeta_p(6s-10) \\ & \times \zeta_p(7s-12)W(p, p^{-s}) \end{aligned}$$

where $W(X, Y)$ is

$$1 + X^4Y^3 + X^8Y^6 + X^9Y^6 - X^9Y^8 - X^{10}Y^8 - X^{14}Y^{11} - X^{18}Y^{14}.$$

$\zeta^{\triangleleft}_{\mathfrak{g}_{37D}}(s)$ is uniform.

3 Functional equation

The local zeta function satisfies the functional equation

$$\zeta^{\triangleleft}_{\mathfrak{g}_{37D},p}(s)\Big|_{p \to p^{-1}} = -p^{21-11s}\zeta^{\triangleleft}_{\mathfrak{g}_{37D},p}(s).$$

4 Abscissa of convergence and order of pole

The abscissa of convergence of $\zeta^{\triangleleft}_{\mathfrak{g}_{37D}}(s)$ is 4, with a simple pole at $s = 4$.

5 Ghost zeta function

The ghost zeta function is the product over all primes of

$$\begin{aligned} & \zeta_p(s)\zeta_p(s-1)\zeta_p(s-2)\zeta_p(s-3)\zeta_p(3s-5)\zeta_p(5s-6)\zeta_p(6s-10)\zeta_p(7s-12) \\ & \times W_1(p, p^{-s})W_2(p, p^{-s}) \end{aligned}$$

where

$$W_1(X,Y) = 1 + X^9Y^6,$$
$$W_2(X,Y) = 1 - X^9Y^8.$$

The ghost is friendly.

6 Natural boundary

$\zeta^{\triangleleft}_{\mathfrak{g}_{37D}}(s)$ has a natural boundary at $\Re(s) = 3/2$, and is of type III.