The zeta function of $F_{2,3}$ counting all subrings

1 Presentation

$F_{2,3}$ has presentation

$$\langle x_1, x_2, x_3, y_1, y_2, y_3 \mid [x_1, x_2] = y_1, [x_1, x_3] = y_2, [x_2, x_3] = y_3 \rangle .$$

$F_{2,3}$ has nilpotency class 2.

2 The local zeta function

The local zeta function was first calculated by Gareth Taylor. It is

$$\begin{aligned}\zeta_{F_{2,3},p}(s) = {} & \zeta_p(s)\zeta_p(s-1)\zeta_p(s-2)\zeta_p(2s-4)\zeta_p(2s-5)\zeta_p(2s-6)\zeta_p(3s-6) \\ & \times \zeta_p(3s-7)\zeta_p(3s-8)\zeta_p(4s-8)^{-1}W(p,p^{-s})\end{aligned}$$

where $W(X,Y)$ is

$$\begin{aligned}&1 + X^3Y^2 + X^4Y^2 + X^5Y^2 - X^4Y^3 - X^5Y^3 - X^6Y^3 - X^7Y^4 - X^9Y^4 \\ &- X^{10}Y^5 - X^{11}Y^5 - X^{12}Y^5 + X^{11}Y^6 + X^{12}Y^6 + X^{13}Y^6 + X^{16}Y^8.\end{aligned}$$

$\zeta_{F_{2,3}}(s)$ is uniform.

3 Functional equation

The local zeta function satisfies the functional equation

$$\zeta_{F_{2,3},p}(s)\big|_{p\to p^{-1}} = p^{15-6s}\zeta_{F_{2,3},p}(s).$$

4 Abscissa of convergence and order of pole

The abscissa of convergence of $\zeta_{F_{2,3}}(s)$ is $7/2$, with a simple pole at $s = 7/2$.

5 Ghost zeta function

The ghost zeta function is the product over all primes of

$$\begin{aligned}&\zeta_p(s)\zeta_p(s-1)\zeta_p(s-2)\zeta_p(2s-5)\zeta_p(2s-6)\zeta_p(3s-6)\zeta_p(3s-7)\zeta_p(3s-8) \\ &\times W_1(p,p^{-s})W_2(p,p^{-s})W_3(p,p^{-s})W_4(p,p^{-s})\end{aligned}$$

where

$$\begin{aligned}W_1(X,Y) &= 1 + X^5Y^2,\\ W_2(X,Y) &= 1 - X^7Y^3,\\ W_3(X,Y) &= -1 - X^4Y^2,\\ W_4(X,Y) &= -1 + X^4Y^3.\end{aligned}$$

The ghost is friendly.

6 Natural boundary

$\zeta_{F_{2,3}}(s)$ has a natural boundary at $\Re(s) = 5/2$, and is of type III.

7 Notes

An excellent calculation by Gareth Taylor gave us this example, the first at class 2 where the abscissa of convergence exceeds the rank of the abelianisation.

Previously, the sign in the functional equation was believed to be negative. This is not the case.