The zeta function of $F_{2,3}$ counting all subrings

1 Presentation

$F_{2,3}$ has presentation

$$\langle x_1, x_2, x_3, y_1, y_2, y_3 \mid [x_1, x_2] = y_1, [x_1, x_3] = y_2, [x_2, x_3] = y_3 \rangle.$$ $F_{2,3}$ has nilpotency class 2.

2 The local zeta function

The local zeta function was first calculated by Gareth Taylor. It is

$$\zeta_{F_{2,3}, p}(s) = \zeta_p(s) \zeta_p(s-1) \zeta_p(s-2) \zeta_p(2s-4) \zeta_p(2s-5) \zeta_p(2s-6) \zeta_p(3s-6)$$

$$\times \zeta_p(3s-7) \zeta_p(3s-8) \zeta_p(4s-8)^{-1} W(p,p^{-s})$$

where $W(X,Y)$ is

$$1 + X^3Y^2 + X^4Y^2 + X^5Y^2 - X^4Y^3 - X^5Y^3 - X^6Y^3 - X^7Y^4 - X^8Y^4 - X^{10}Y^5 - X^{11}Y^5 - X^{12}Y^5 + X^{11}Y^6 + X^{12}Y^6 + X^{13}Y^6 + X^{16}Y^8.$$

$\zeta_{F_{2,3},p}(s)$ is uniform.

3 Functional equation

The local zeta function satisfies the functional equation

$$\zeta_{F_{2,3},p}(s)\bigg|_{p \rightarrow p^{-1}} = p^{15-6s} \zeta_{F_{2,3},p}(s).$$

4 Abscissa of convergence and order of pole

The abscissa of convergence of $\zeta_{F_{2,3},p}(s)$ is $7/2$, with a simple pole at $s = 7/2$.

5 Ghost zeta function

The ghost zeta function is the product over all primes of

$$\zeta_p(s) \zeta_p(s-1) \zeta_p(s-2) \zeta_p(2s-5) \zeta_p(3s-6) \zeta_p(3s-7) \zeta_p(3s-8)$$

$$\times W_1(p,p^{-s}) W_2(p,p^{-s}) W_3(p,p^{-s}) W_4(p,p^{-s})$$
where
\begin{align*}
W_1(X,Y) &= 1 + X^5Y^2, \\
W_2(X,Y) &= 1 - X^7Y^3, \\
W_3(X,Y) &= -1 - X^4Y^2, \\
W_4(X,Y) &= -1 + X^4Y^3.
\end{align*}

The ghost is friendly.

6 Natural boundary
\(\zeta_{F_{2,3}}(s)\) has a natural boundary at \(\Re(s) = 5/2\), and is of type III.

7 Notes
An excellent calculation by Gareth Taylor gave us this example, the first at class 2 where the abscissa of convergence exceeds the rank of the abelianisation.

Previously, the sign in the functional equation was believed to be negative. This is not the case.