The zeta function of $F_{3,2}$

counting ideals

1 Presentation

$F_{3,2}$ has presentation

$\langle x_1, x_2, y_1, z_1, z_2 \mid [x_1, x_2] = y_1, [x_1, y_1] = z_1, [x_2, y_1] = z_2 \rangle$.

$F_{3,2}$ has nilpotency class 3.

2 The local zeta function

The local zeta function was first calculated by Luke Woodward. It is

$\zeta_{F_{3,2},p}(s) = \zeta_p(s) \zeta_p(s - 1) \zeta_p(3s - 2) \zeta_p(4s - 3) \zeta_p(5s - 4) \zeta_p(7s - 6)$

$\times W(p, p^{-s})$

where $W(X, Y)$ is

$1 + X^2 Y^4 - X^2 Y^5 - X^4 Y^7 - X^6 Y^9 - X^8 Y^{11} + X^8 Y^{12} + X^{10} Y^{16}$.

$\zeta_{F_{3,2}}(s)$ is uniform.

3 Functional equation

The local zeta function satisfies the functional equation

$\left. \zeta_{F_{3,2},p}(s) \right|_{p \to p^{-1}} = -p^{10 - 10s} \zeta_{F_{3,2},p}(s)$.

4 Abscissa of convergence and order of pole

The abscissa of convergence of $\zeta_{F_{3,2}}(s)$ is 2, with a simple pole at $s = 2$.

5 Ghost zeta function

The ghost zeta function is the product over all primes of

$\zeta_p(s) \zeta_p(s - 1) \zeta_p(3s - 2) \zeta_p(4s - 3) \zeta_p(5s - 4) \zeta_p(7s - 6) W_1(p, p^{-s}) W_2(p, p^{-s})$

where

$W_1(X, Y) = 1 - X^8 Y^{11}$,

$W_2(X, Y) = -1 + X^2 Y^5$.

The ghost is friendly.
6 Natural boundary

\( \zeta_{F_{3,2}}(s) \) has a natural boundary at \( \Re(s) = 8/11 \), and is of type III.