The zeta function of $F_{3,2} \times \mathbb{Z}$

counting ideals

1 Presentation

$F_{3,2} \times \mathbb{Z}$ has presentation

\[ \langle x_1, x_2, y, a, z_1, z_2 \mid [x_1, x_2] = y, [x_1, y] = z_1, [x_2, y] = z_2 \rangle. \]

$F_{3,2} \times \mathbb{Z}$ has nilpotency class 3.

2 The local zeta function

The local zeta function was first calculated by Luke Woodward. It is

\[ \zeta_{F_{3,2} \times \mathbb{Z}, p}(s) = \zeta_p(s)\zeta_p(s - 1)\zeta_p(s - 2)\zeta_p(3s - 3)\zeta_p(4s - 4)\zeta_p(5s - 5)\zeta_p(5s - 6) \times \zeta_p(7s - 8)W(p, p^{-s}) \]

where $W(X, Y)$ is

\[ 1 + X^3Y^4 - X^3Y^5 - X^6Y^7 - X^8Y^9 - X^{11}Y^{11} + X^{11}Y^{12} + X^{14}Y^{16}. \]

$\zeta_{F_{3,2} \times \mathbb{Z}}(s)$ is uniform.

3 Functional equation

The local zeta function satisfies the functional equation

\[ \zeta_{F_{3,2} \times \mathbb{Z}, p}(s) \big|_{p \to p^{-1}} = p^{15 - 11s} \zeta_{F_{3,2} \times \mathbb{Z}}(s). \]

4 Abscissa of convergence and order of pole

The abscissa of convergence of $\zeta_{F_{3,2} \times \mathbb{Z}}(s)$ is 3, with a simple pole at $s = 3$.

5 Ghost zeta function

The ghost zeta function is the product over all primes of

\[ \zeta_p(s)\zeta_p(s - 1)\zeta_p(s - 2)\zeta_p(3s - 3)\zeta_p(4s - 4)\zeta_p(5s - 5)\zeta_p(5s - 6)\zeta_p(7s - 8) \times W_1(p, p^{-s})W_2(p, p^{-s}) \]
where

\[ W_1(X,Y) = 1 - X^{11}Y^{11}, \]
\[ W_2(X,Y) = -1 + X^3Y^5. \]

The ghost is friendly.

6 Natural boundary

\( \zeta_{F_{3,2\times2}}(s) \) has a natural boundary at \( \Re(s) = 1 \), and is of type II.