The zeta function of $F_{3,2} \times \mathbb{Z}$ counting ideals

1 Presentation

$F_{3,2} \times \mathbb{Z}$ has presentation

$$\langle x_1, x_2, y, a, z_1, z_2 \mid [x_1, x_2] = y, [x_1, y] = z_1, [x_2, y] = z_2 \rangle .$$

$F_{3,2} \times \mathbb{Z}$ has nilpotency class 3.

2 The local zeta function

The local zeta function was first calculated by Luke Woodward. It is

$$\begin{aligned}\zeta^{\triangleleft}_{F_{3,2}\times\mathbb{Z},p}(s) = \zeta_p(s)\zeta_p(s-1)\zeta_p(s-2)\zeta_p(3s-3)\zeta_p(4s-4)\zeta_p(5s-5)\zeta_p(5s-6)\\ \times \zeta_p(7s-8)W(p,p^{-s})\end{aligned}$$

where $W(X,Y)$ is

$$1 + X^3Y^4 - X^3Y^5 - X^6Y^7 - X^8Y^9 - X^{11}Y^{11} + X^{11}Y^{12} + X^{14}Y^{16}.$$

$\zeta^{\triangleleft}_{F_{3,2}\times\mathbb{Z}}(s)$ is uniform.

3 Functional equation

The local zeta function satisfies the functional equation

$$\zeta^{\triangleleft}_{F_{3,2}\times\mathbb{Z},p}(s)\Big|_{p\to p^{-1}} = p^{15-11s}\zeta^{\triangleleft}_{F_{3,2}\times\mathbb{Z},p}(s).$$

4 Abscissa of convergence and order of pole

The abscissa of convergence of $\zeta^{\triangleleft}_{F_{3,2}\times\mathbb{Z}}(s)$ is 3, with a simple pole at $s = 3$.

5 Ghost zeta function

The ghost zeta function is the product over all primes of

$$\begin{aligned}&\zeta_p(s)\zeta_p(s-1)\zeta_p(s-2)\zeta_p(3s-3)\zeta_p(4s-4)\zeta_p(5s-5)\zeta_p(5s-6)\zeta_p(7s-8)\\ &\times W_1(p,p^{-s})W_2(p,p^{-s})\end{aligned}$$

where

$$W_1(X,Y) = 1 - X^{11}Y^{11},$$
$$W_2(X,Y) = -1 + X^3Y^5.$$

The ghost is friendly.

6 Natural boundary

$\zeta^{\triangleleft}_{F_{3,2}\times\mathbb{Z}}(s)$ has a natural boundary at $\Re(s) = 1$, and is of type II.