The zeta function of Fil_4 counting ideals

1 Presentation

Fil_4 has presentation

$$\langle z, x_1, x_2, x_3, x_4 \mid [z, x_1] = x_2, [z, x_2] = x_3, [z, x_3] = x_4, [x_1, x_2] = x_4 \rangle .$$

Fil_4 has nilpotency class 4.

2 The local zeta function

The local zeta function was first calculated by Luke Woodward. It is

$$\begin{aligned}\zeta^{\triangleleft}_{\mathrm{Fil}_4,p}(s) = {} & \zeta_p(s)\zeta_p(s-1)\zeta_p(3s-2)\zeta_p(5s-2)\zeta_p(7s-4)\zeta_p(8s-5)\zeta_p(9s-6)\\ & \times \zeta_p(10s-6)\zeta_p(12s-7)W(p, p^{-s})\end{aligned}$$

where $W(X, Y)$ is

$$\begin{aligned}&1 + X^2Y^4 - X^2Y^5 + X^3Y^5 - X^2Y^6 + X^3Y^6 - X^3Y^7 - X^5Y^9 - X^5Y^{10}\\
&- X^6Y^{11} - X^6Y^{12} + X^6Y^{13} - X^7Y^{13} - X^8Y^{13} - X^8Y^{14} + X^7Y^{15} + X^8Y^{15}\\
&- 2X^9Y^{15} + X^8Y^{17} + X^9Y^{17} - X^{10}Y^{17} + X^9Y^{19} + X^{10}Y^{19} + X^{11}Y^{20}\\
&+ 2X^{11}Y^{21} - X^{11}Y^{22} + 2X^{12}Y^{22} + 2X^{13}Y^{23} - X^{13}Y^{24} + X^{14}Y^{24}\\
&- X^{13}Y^{25} + X^{14}Y^{25} + X^{15}Y^{25} - 2X^{14}Y^{27} + 2X^{15}Y^{27} - 2X^{15}Y^{28}\\
&+ X^{16}Y^{28} - X^{15}Y^{29} - X^{16}Y^{29} + X^{17}Y^{29} - 2X^{17}Y^{30} + X^{18}Y^{30} - X^{18}Y^{31}\\
&- X^{18}Y^{32} - X^{18}Y^{33} - X^{20}Y^{35} + X^{20}Y^{36} - X^{21}Y^{36} + X^{20}Y^{37} - X^{21}Y^{37}\\
&+ X^{21}Y^{38} + X^{23}Y^{42}.\end{aligned}$$

$\zeta^{\triangleleft}_{\mathrm{Fil}_4}(s)$ is uniform.

3 Functional equation

The local zeta function satisfies no functional equation.

4 Abscissa of convergence and order of pole

The abscissa of convergence of $\zeta^{\triangleleft}_{\mathrm{Fil}_4}(s)$ is 2, with a simple pole at $s = 2$.

5 Ghost zeta function

The ghost zeta function is the product over all primes of

$$
\begin{aligned}
&\zeta_p(s)\zeta_p(s-1)\zeta_p(3s-2)\zeta_p(5s-2)\zeta_p(7s-4)\zeta_p(8s-5)\zeta_p(9s-6)\zeta_p(10s-6) \\
&\times \zeta_p(12s-7)W_1(p,p^{-s})W_2(p,p^{-s})W_3(p,p^{-s})W_4(p,p^{-s})
\end{aligned}
$$

where

$$
\begin{aligned}
W_1(X,Y) &= 1 - X^8Y^{13}, \\
W_2(X,Y) &= -1 + X^{10}Y^{17}, \\
W_3(X,Y) &= 1 - X^3Y^6, \\
W_4(X,Y) &= -1 + X^2Y^6.
\end{aligned}
$$

The ghost is friendly.

6 Natural boundary

$\zeta^{\triangleleft}_{\mathrm{Fil}_4}(s)$ has a natural boundary at $\Re(s) = 8/13$, and is of type III.

7 Notes

This was the second local ideal zeta function with no functional equation. It is also the smallest such, in terms of dimension.