The zeta function of $\mathrm{Fil}_4 \times \mathbb{Z}$ counting ideals

1 Presentation

$\mathrm{Fil}_4 \times \mathbb{Z}$ has presentation

$$\langle z, x_1, x_2, a, x_3, x_4 \mid [z, x_1] = x_2, [z, x_2] = x_3, [z, x_3] = x_4, [x_1, x_2] = x_4 \rangle .$$

$\mathrm{Fil}_4 \times \mathbb{Z}$ has nilpotency class 4.

2 The local zeta function

The local zeta function was first calculated by Luke Woodward. It is

$$\begin{aligned}\zeta^{\triangleleft}_{\mathrm{Fil}_4 \times \mathbb{Z}, p}(s) = \zeta_p(s)\zeta_p(s-1)\zeta_p(s-2)\zeta_p(3s-3)\zeta_p(5s-3)\zeta_p(7s-5)\zeta_p(8s-7) \\ \times \zeta_p(9s-8)\zeta_p(10s-8)\zeta_p(12s-9)W(p, p^{-s})\end{aligned}$$

where $W(X, Y)$ is

$$\begin{aligned}&1 + X^3Y^4 - X^3Y^5 + X^4Y^5 - X^3Y^6 + X^4Y^6 - X^4Y^7 - X^7Y^9 - X^7Y^{10} \\ &- X^8Y^{11} - X^8Y^{12} + X^8Y^{13} - X^9Y^{13} - X^{11}Y^{13} - X^{11}Y^{14} + X^{10}Y^{15} \\ &+ X^{11}Y^{15} - 2X^{12}Y^{15} + X^{11}Y^{17} + X^{12}Y^{17} - X^{13}Y^{17} + X^{12}Y^{19} + X^{14}Y^{19} \\ &+ X^{15}Y^{20} + 2X^{15}Y^{21} - X^{15}Y^{22} + 2X^{16}Y^{22} + X^{17}Y^{23} + X^{18}Y^{23} - X^{18}Y^{24} \\ &+ X^{19}Y^{24} - X^{18}Y^{25} + X^{19}Y^{25} + X^{20}Y^{25} - 2X^{19}Y^{27} + 2X^{20}Y^{27} \\ &- 2X^{20}Y^{28} + X^{21}Y^{28} - X^{20}Y^{29} - X^{22}Y^{29} + X^{23}Y^{29} - 2X^{23}Y^{30} + X^{24}Y^{30} \\ &- X^{24}Y^{31} - X^{24}Y^{32} - X^{24}Y^{33} - X^{27}Y^{35} + X^{27}Y^{36} - X^{28}Y^{36} + X^{27}Y^{37} \\ &- X^{28}Y^{37} + X^{28}Y^{38} + X^{31}Y^{42}.\end{aligned}$$

$\zeta^{\triangleleft}_{\mathrm{Fil}_4 \times \mathbb{Z}}(s)$ is uniform.

3 Functional equation

The local zeta function satisfies no functional equation.

4 Abscissa of convergence and order of pole

The abscissa of convergence of $\zeta^{\triangleleft}_{\mathrm{Fil}_4 \times \mathbb{Z}}(s)$ is 3, with a simple pole at $s = 3$.

5 Ghost zeta function

The ghost zeta function is the product over all primes of

$$\zeta_p(s)\zeta_p(s-1)\zeta_p(s-2)\zeta_p(3s-3)\zeta_p(5s-3)\zeta_p(7s-5)\zeta_p(8s-7)\zeta_p(9s-8)$$
$$\times\,\zeta_p(10s-8)\zeta_p(12s-9)W_1(p,p^{-s})W_2(p,p^{-s})W_3(p,p^{-s})W_4(p,p^{-s})$$

where

$$W_1(X,Y) = 1 - X^{11}Y^{13},$$
$$W_2(X,Y) = -1 + X^{13}Y^{17},$$
$$W_3(X,Y) = 1 - X^{4}Y^{6},$$
$$W_4(X,Y) = -1 + X^{3}Y^{6}.$$

The ghost is friendly.

6 Natural boundary

$\zeta^{\triangleleft}_{\mathrm{Fil}_4\times\mathbb{Z}}(s)$ has a natural boundary at $\Re(s) = 11/13$, and is of type III.