The zeta function of G_3 counting all subrings

1 Presentation

G_3 has presentation

$$\langle z, x_1, x_2, y_1, y_2 \mid [z, x_1] = y_1, [z, x_2] = y_2 \rangle .$$

G_3 has nilpotency class 2.

2 The local zeta function

The local zeta function was first calculated by Dermot Grenham. It is

$$\zeta_{G_3,p}(s) = \zeta_p(s)\zeta_p(s-1)\zeta_p(s-2)\zeta_p(2s-4)\zeta_p(2s-5)\zeta_p(3s-6)W(p, p^{-s})$$

where $W(X, Y)$ is

$$1 + X^3Y^2 + X^4Y^2 - X^4Y^3 - X^5Y^3 - X^8Y^5.$$

$\zeta_{G_3}(s)$ is uniform.

3 Functional equation

The local zeta function satisfies the functional equation

$$\zeta_{G_3,p}(s)|_{p\to p^{-1}} = -p^{10-5s}\zeta_{G_3,p}(s).$$

4 Abscissa of convergence and order of pole

The abscissa of convergence of $\zeta_{G_3}(s)$ is 3, with a double pole at $s = 3$.

5 Ghost zeta function

The ghost zeta function is the product over all primes of

$$\zeta_p(s)\zeta_p(s-1)\zeta_p(s-2)\zeta_p(2s-4)\zeta_p(2s-5)\zeta_p(3s-6)W_1(p, p^{-s})W_2(p, p^{-s})$$

where

$$W_1(X, Y) = 1 + X^4Y^2,$$
$$W_2(X, Y) = 1 - X^4Y^3.$$

The ghost is friendly.

6 Natural boundary

$\zeta_{G_3}(s)$ has a natural boundary at $\Re(s) = 2$, and is of type II.