The zeta function of $G_3 \times Q_5$ counting ideals

1 Presentation

$G_3 \times Q_5$ has presentation

$$\left\langle c, x_1, x_2, x_4, a_1, a_2, x_3, x_5, b_1, b_2 \;\middle|\; \begin{array}{c} [x_1, x_2] = x_3, [x_1, x_3] = x_5, [x_2, x_4] = x_5, \\ [c, a_1] = b_1, [c, a_2] = b_2 \end{array} \right\rangle.$$

$G_3 \times Q_5$ has nilpotency class 3.

2 The local zeta function

The local zeta function was first calculated by Luke Woodward. It is

$$\begin{aligned} \zeta^{\triangleleft}_{G_3 \times Q_5, p}(s) = {} & \zeta_p(s)\zeta_p(s-1)\zeta_p(s-2)\zeta_p(s-3)\zeta_p(s-4)\zeta_p(s-5)\zeta_p(3s-6) \\ & \times \zeta_p(3s-7)\zeta_p(5s-7)\zeta_p(5s-8)\zeta_p(5s-12)\zeta_p(7s-9) \\ & \times \zeta_p(7s-14)\zeta_p(9s-15)\zeta_p(11s-16)W(p, p^{-s}) \end{aligned}$$

where $W(X, Y)$ is

$$\begin{aligned} & 1 + X^{6}Y^{3} - X^{6}Y^{5} - X^{7}Y^{7} - X^{12}Y^{7} - X^{14}Y^{8} - X^{13}Y^{9} - X^{15}Y^{10} - X^{20}Y^{10} \\ & + X^{13}Y^{11} - X^{14}Y^{11} - X^{15}Y^{11} + X^{14}Y^{12} + X^{15}Y^{12} - X^{16}Y^{12} + X^{20}Y^{12} \\ & - X^{21}Y^{12} + X^{19}Y^{14} + 2X^{21}Y^{14} - X^{22}Y^{14} - X^{23}Y^{14} + X^{23}Y^{15} + X^{26}Y^{15} \\ & + 2X^{22}Y^{16} + X^{26}Y^{16} + X^{27}Y^{16} - X^{28}Y^{16} - X^{26}Y^{17} + X^{27}Y^{17} + X^{28}Y^{17} \\ & + X^{29}Y^{17} + X^{23}Y^{18} + X^{28}Y^{18} - X^{27}Y^{19} + X^{28}Y^{19} + X^{30}Y^{19} + X^{35}Y^{19} \\ & + X^{29}Y^{20} - X^{28}Y^{21} + X^{31}Y^{21} - X^{33}Y^{21} + X^{36}Y^{21} - X^{35}Y^{22} - X^{29}Y^{23} \\ & - X^{34}Y^{23} - X^{36}Y^{23} + X^{37}Y^{23} - X^{36}Y^{24} - X^{41}Y^{24} - X^{35}Y^{25} - X^{36}Y^{25} \\ & - X^{37}Y^{25} + X^{38}Y^{25} + X^{36}Y^{26} - X^{37}Y^{26} - X^{38}Y^{26} - 2X^{42}Y^{26} - X^{38}Y^{27} \\ & - X^{41}Y^{27} + X^{41}Y^{28} + X^{42}Y^{28} - 2X^{43}Y^{28} - X^{45}Y^{28} + X^{43}Y^{30} - X^{44}Y^{30} \\ & + X^{48}Y^{30} - X^{49}Y^{30} - X^{50}Y^{30} + X^{49}Y^{31} + X^{50}Y^{31} - X^{51}Y^{31} + X^{44}Y^{32} \\ & + X^{49}Y^{32} + X^{51}Y^{33} + X^{50}Y^{34} + X^{52}Y^{35} + X^{57}Y^{35} + X^{58}Y^{37} - X^{58}Y^{39} \\ & - X^{64}Y^{42}. \end{aligned}$$

$\zeta^{\triangleleft}_{G_3 \times Q_5}(s)$ is uniform.

3 Functional equation

The local zeta function satisfies the functional equation

$$\zeta^{\triangleleft}_{G_3\times Q_5,p}(s)\Big|_{p\to p^{-1}} = p^{45-19s}\zeta^{\triangleleft}_{G_3\times Q_5,p}(s).$$

4 Abscissa of convergence and order of pole

The abscissa of convergence of $\zeta^{\triangleleft}_{G_3\times Q_5}(s)$ is 6, with a simple pole at $s = 6$.

5 Ghost zeta function

The ghost zeta function is the product over all primes of

$$\begin{aligned}&\zeta_p(s)\zeta_p(s-1)\zeta_p(s-2)\zeta_p(s-3)\zeta_p(s-4)\zeta_p(s-5)\zeta_p(3s-6)\zeta_p(3s-7)\\ &\times \zeta_p(5s-7)\zeta_p(5s-8)\zeta_p(5s-12)\zeta_p(7s-9)\zeta_p(7s-14)\zeta_p(9s-15)\zeta_p(11s-16)\\ &\times W_1(p,p^{-s})W_2(p,p^{-s})W_3(p,p^{-s})W_4(p,p^{-s})\end{aligned}$$

where

$$\begin{aligned}W_1(X,Y) &= 1 + X^6Y^3 - X^{20}Y^{10},\\ W_2(X,Y) &= -1 + X^{15}Y^9,\\ W_3(X,Y) &= 1 + X^{22}Y^{16},\\ W_4(X,Y) &= 1 - X^7Y^7.\end{aligned}$$

The ghost is unfriendly.

6 Natural boundary

$\zeta^{\triangleleft}_{G_3\times Q_5}(s)$ has a natural boundary at $\Re(s) = 2$, and is of type I.