The zeta function of $G_4$
counting all subrings

1 Presentation

$G_4$ has presentation

$$\langle z, x_1, x_2, x_3, y_1, y_2, y_3 \mid [z, x_1] = y_1, [z, x_2] = y_2, [z, x_3] = y_3 \rangle.$$

$G_4$ has nilpotency class 2.

2 The local zeta function

The local zeta function was first calculated by Dermot Grenham. It is

$$\zeta_{G_4, p}(s) = \zeta_p(s)\zeta_p(s - 1)\zeta_p(s - 2)\zeta_p(2s - 5)\zeta_p(2s - 6)\zeta_p(2s - 7) \times \zeta_p(3s - 10)\zeta_p(4s - 12)W(p, p^{-s})\times W(X, Y)$$

where $W(X, Y)$ is

$$1 + X^4Y^2 + X^5Y^2 - X^6Y^3 - X^6Y^3 - X^7Y^3 + X^8Y^3 + X^9Y^3 - X^{10}Y^4 - X^{11}Y^4 - X^{14}Y^6 - X^{15}Y^6 - X^{16}Y^6 + X^{16}Y^7 + X^{17}Y^7 - X^{18}Y^7 - X^{19}Y^7 - X^{20}Y^7 + X^{20}Y^8 + X^{21}Y^8 + X^{25}Y^{10}.$$

$\zeta_{G_4(s)}$ is uniform.

3 Functional equation

The local zeta function satisfies the functional equation

$$\zeta_{G_4, p}(s) \big|_{p \to p^{-1}} = -p^{21-7s}\zeta_{G_4, p}(s).$$

4 Abscissa of convergence and order of pole

The abscissa of convergence of $\zeta_{G_4}(s)$ is 4, with a double pole at $s = 4$.

5 Ghost zeta function

The ghost zeta function is the product over all primes of

$$\zeta_p(s)\zeta_p(s - 1)\zeta_p(s - 2)\zeta_p(s - 3)\zeta_p(2s - 5)\zeta_p(2s - 6)\zeta_p(2s - 7)\zeta_p(3s - 10) \times \zeta_p(4s - 12)W_1(p, p^{-s})W_2(p, p^{-s})W_3(p, p^{-s})$$
where

\[ W_1(X,Y) = 1 + X^6Y^2 + X^9Y^3, \]
\[ W_2(X,Y) = 1 - X^{11}Y^4, \]
\[ W_3(X,Y) = -1 + X^5Y^3. \]

The ghost is unfriendly.

6 Natural boundary

\( \zeta_{G_4}(s) \) has a natural boundary at \( \Re(s) = 3 \), and is of type I.