The zeta function of G_4 counting all subrings

1 Presentation

G_4 has presentation

$$\langle z, x_1, x_2, x_3, y_1, y_2, y_3 \mid [z, x_1] = y_1, [z, x_2] = y_2, [z, x_3] = y_3 \rangle .$$

G_4 has nilpotency class 2.

2 The local zeta function

The local zeta function was first calculated by Dermot Grenham. It is

$$\begin{aligned}\zeta_{G_4,p}(s) = {} & \zeta_p(s)\zeta_p(s-1)\zeta_p(s-2)\zeta_p(s-3)\zeta_p(2s-5)\zeta_p(2s-6)\zeta_p(2s-7) \\ & \times \zeta_p(3s-10)\zeta_p(4s-12)W(p, p^{-s})\end{aligned}$$

where $W(X, Y)$ is

$$\begin{aligned}& 1 + X^4Y^2 + X^5Y^2 + X^6Y^2 - X^5Y^3 - X^6Y^3 - X^7Y^3 + X^8Y^3 + X^9Y^3 \\ & - X^9Y^4 - X^{10}Y^4 - X^{11}Y^4 - X^{14}Y^6 - X^{15}Y^6 - X^{16}Y^6 + X^{16}Y^7 + X^{17}Y^7 \\ & - X^{18}Y^7 - X^{19}Y^7 - X^{20}Y^7 + X^{19}Y^8 + X^{20}Y^8 + X^{21}Y^8 + X^{25}Y^{10}.\end{aligned}$$

$\zeta_{G_4}(s)$ is uniform.

3 Functional equation

The local zeta function satisfies the functional equation

$$\zeta_{G_4,p}(s)\big|_{p \to p^{-1}} = -p^{21-7s}\zeta_{G_4,p}(s).$$

4 Abscissa of convergence and order of pole

The abscissa of convergence of $\zeta_{G_4}(s)$ is 4, with a double pole at $s = 4$.

5 Ghost zeta function

The ghost zeta function is the product over all primes of

$$\begin{aligned}& \zeta_p(s)\zeta_p(s-1)\zeta_p(s-2)\zeta_p(s-3)\zeta_p(2s-5)\zeta_p(2s-6)\zeta_p(2s-7)\zeta_p(3s-10) \\ & \times \zeta_p(4s-12)W_1(p, p^{-s})W_2(p, p^{-s})W_3(p, p^{-s})\end{aligned}$$

where

$$\begin{aligned}
W_1(X,Y) &= 1 + X^6Y^2 + X^9Y^3, \\
W_2(X,Y) &= 1 - X^{11}Y^4, \\
W_3(X,Y) &= -1 + X^5Y^3.
\end{aligned}$$

The ghost is unfriendly.

6 Natural boundary

$\zeta_{G_4}(s)$ has a natural boundary at $\Re(s) = 3$, and is of type I.