The zeta function of $G_5$

counting ideals

1  Presentation

$G_5$ has presentation

$$\left\langle z, x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4 \mid [z, x_1] = y_1, [z, x_2] = y_2, [z, x_3] = y_3, [z, x_4] = y_4 \right\rangle.$$ 

$G_5$ has nilpotency class 2.

2  The local zeta function

The local zeta function was first calculated by Dermot Grenham. It is

$$\zeta_{G_5,p}(s) = \zeta_p(s)\zeta_p(s-1)\zeta_p(s-2)\zeta_p(s-3)\zeta_p(s-4)\zeta_p(5s-8)\zeta_p(5s-14) \times \zeta_p(7s-18)\zeta_p(9s-20)W(p, p^{-s})$$

where $W(X,Y)$ is

$$1 + X^5Y^3 + X^6Y^3 + X^7Y^3 + X^{10}Y^5 + X^{11}Y^5 + 2X^{12}Y^5 + X^{13}Y^5 + X^{15}Y^7 + X^{16}Y^7 + X^{17}Y^7 + X^{18}Y^8 + X^{19}Y^8 + X^{21}Y^{10} + \ldots$$

$\zeta_{G_5}(s)$ is uniform.

3  Functional equation

The local zeta function satisfies the functional equation

$$\zeta_{G_5,p}(s) \mid_{p \to p^{-1}} = -p^{36-14s} \zeta_{G_5,p}(s).$$

4  Abscissa of convergence and order of pole

The abscissa of convergence of $\zeta_{G_5}(s)$ is 5, with a simple pole at $s = 5$.

5  Ghost zeta function

The ghost zeta function is the product over all primes of

$$\zeta_p(s)\zeta_p(s-1)\zeta_p(s-2)\zeta_p(s-3)\zeta_p(s-4)\zeta_p(3s-8)\zeta_p(5s-14)\zeta_p(7s-18) \times \zeta_p(9s-20)W_1(p, p^{-s})W_2(p, p^{-s})W_3(p, p^{-s})$$
where

\[ W_1(X,Y) = 1 + X^{13}Y^5, \]
\[ W_2(X,Y) = 1 + X^{16}Y^7, \]
\[ W_3(X,Y) = 1 + X^5Y^3. \]

The ghost is friendly.

6 Natural boundary

\( \zeta_{G_5}^\infty(s) \) has a natural boundary at \( \Re(s) = 13/5 \), and is of type III.