The zeta function of G_5 counting all subrings

1 Presentation

G_5 has presentation

$$\left\langle z, x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4 \;\middle|\; \begin{array}{c} [z, x_1] = y_1, [z, x_2] = y_2, \\ [z, x_3] = y_3, [z, x_4] = y_4 \end{array} \right\rangle.$$

G_5 has nilpotency class 2.

2 The local zeta function

The local zeta function was first calculated by Dermot Grenham. It is

$$\begin{aligned}\zeta_{G_5,p}(s) = {} & \zeta_p(s)\zeta_p(s-1)\zeta_p(s-3)\zeta_p(s-4)\zeta_p(2s-6)\zeta_p(2s-8)\zeta_p(2s-9) \\ & \times \zeta_p(3s-14)\zeta_p(4s-18)\zeta_p(5s-20)W(p,p^{-s})\end{aligned}$$

where $W(X,Y)$ is

$$\begin{aligned}
&1 + X^2Y + X^4Y^2 + X^5Y^2 + X^6Y^2 + 2X^7Y^2 + X^8Y^2 + X^9Y^3 + 2X^{10}Y^3 \\
&+ X^{11}Y^3 + 2X^{12}Y^3 + X^{13}Y^3 + X^{12}Y^4 + 2X^{14}Y^4 + 2X^{15}Y^4 + X^{16}Y^4 \\
&+ X^{17}Y^4 + 2X^{17}Y^5 + X^{18}Y^5 + 2X^{19}Y^5 + X^{20}Y^5 - X^{18}Y^6 - X^{20}Y^6 \\
&+ X^{21}Y^6 + 2X^{22}Y^6 + 2X^{23}Y^6 + 2X^{24}Y^6 + X^{25}Y^6 - X^{22}Y^7 - 2X^{23}Y^7 \\
&- 2X^{24}Y^7 - 2X^{25}Y^7 - X^{26}Y^7 + X^{27}Y^7 + X^{29}Y^7 - X^{27}Y^8 - 2X^{28}Y^8 \\
&- X^{29}Y^8 - 2X^{30}Y^8 - X^{30}Y^9 - X^{31}Y^9 - 2X^{32}Y^9 - 2X^{33}Y^9 - X^{35}Y^9 \\
&- X^{34}Y^{10} - 2X^{35}Y^{10} - X^{36}Y^{10} - 2X^{37}Y^{10} - X^{38}Y^{10} - X^{39}Y^{11} \\
&- 2X^{40}Y^{11} - X^{41}Y^{11} - X^{42}Y^{11} - X^{43}Y^{11} - X^{45}Y^{12} - X^{47}Y^{13}.
\end{aligned}$$

$\zeta_{G_5}(s)$ is uniform.

3 Functional equation

The local zeta function satisfies the functional equation

$$\zeta_{G_5,p}(s)\big|_{p\to p^{-1}} = -p^{36-9s}\zeta_{G_5,p}(s).$$

4 Abscissa of convergence and order of pole

The abscissa of convergence of $\zeta_{G_5}(s)$ is 5, with a triple pole at $s = 5$.

5 Ghost zeta function

The ghost zeta function is the product over all primes of

$$\zeta_p(s)\zeta_p(s-1)\zeta_p(s-3)\zeta_p(s-4)\zeta_p(2s-6)\zeta_p(2s-8)\zeta_p(2s-9)\zeta_p(3s-14)$$
$$\times\, \zeta_p(4s-18)\zeta_p(5s-20)W_1(p,p^{-s})W_2(p,p^{-s})W_3(p,p^{-s})W_4(p,p^{-s})$$

where

$$\begin{aligned}
W_1(X,Y) &= 1 + X^{13}Y^3,\\
W_2(X,Y) &= 1 + X^4Y + X^{12}Y^3 + X^{16}Y^4,\\
W_3(X,Y) &= 1 - X^{14}Y^4,\\
W_4(X,Y) &= -1 - X^2Y - X^4Y^2.
\end{aligned}$$

The ghost is friendly.

6 Natural boundary

$\zeta_{G_5}(s)$ has a natural boundary at $\Re(s) = 13/3$, and is of type III.