The zeta function of $H(\mathcal{O}_{K_2})$ counting ideals

1 Presentation

Let K_2 be a quadratic number field, and $\mathcal{O}_{K_2}$ its ring of integers. $H(\mathcal{O}_{K_2})$, the Heisenberg Lie ring over $\mathcal{O}_{K_2}$, has presentation

$$\left\langle x_1, x_2, x_3, x_4, x_5, x_6 \;\middle|\; \begin{array}{c} [x_1, x_4] = x_6, [x_1, x_3] = x_5, [x_2, x_3] = x_6, \\ [x_2, x_4] = \alpha x_5 + \beta x_6 \end{array} \right\rangle.$$

where $K_2 = \mathbb{Q}(\gamma)$ for γ squarefree, and

$$\alpha x_5 + \beta x_6 = \begin{cases} \gamma x_5 & \text{if } \gamma \equiv 2, 3(\pmod 4) \\ \frac{1}{4}(\gamma - 1)x_5 + x_6 & \text{if } \gamma \equiv 1(\pmod 4) \end{cases}.$$

$H(\mathcal{O}_{K_2})$ has nilpotency class 2.

2 The local zeta function

The local zeta functions were first calculated by Grunewald, Segal & Smith. The local zeta functions depend on the behaviour of the prime p in the ring of integers of $\mathbb{Q}(\sqrt{\gamma})$. For p ramified,

$$\zeta^{\triangleleft}_{H(\mathcal{O}_{K_2}),p}(s) = \zeta_p(s)\zeta_p(s-1)\zeta_p(s-2)\zeta_p(s-3)\zeta_p(3s-4)\zeta_p(5s-5).$$

For inert primes p, it is

$$\zeta^{\triangleleft}_{H(\mathcal{O}_{K_2}),p}(s) = \zeta_p(s)\zeta_p(s-1)\zeta_p(s-2)\zeta_p(s-3)\zeta_p(5s-5)\zeta_p(6s-8)(1+p^{4-5s}).$$

For split primes p, it is

$$\zeta^{\triangleleft}_{H(\mathcal{O}_{K_2}),p}(s) = \zeta_p(s)\zeta_p(s-1)\zeta_p(s-2)\zeta_p(3s-4)^2\zeta_p(5s-5)\zeta_p(5s-4)^{-1}$$

$\zeta^{\triangleleft}_{H(\mathcal{O}_{K_2})}(s)$ is finitely uniform.

3 Functional equation

For p split or inert, the local zeta function satisfies the functional equation

$$\zeta^{\triangleleft}_{H(\mathcal{O}_{K_2}),p}(s)|_{p\to p^{-1}} = p^{15-10s}\zeta^{\triangleleft}_{H(\mathcal{O}_{K_2}),p}(s).$$

The p ramified case also satisfies a functional equation,

$$\zeta^{\triangleleft}_{H(\mathcal{O}_{K_2}),p}(s)|_{p\to p^{-1}} = p^{15-12s}\zeta^{\triangleleft}_{H(\mathcal{O}_{K_2}),p}(s),$$

but this is unlikely to be of any significance in general. The functional equation arises due to the simple form of the rational function.

4 Abscissa of convergence and order of pole

The abscissa of convergence of $\zeta^{\triangleleft}_{H(\mathcal{O}_{K_2})}(s)$ is 4, with a simple pole at $s = 4$.

5 Ghost zeta function

The ghost zeta function is unknown, since the local zeta functions vary with the prime.

6 Natural boundary

Since a factor of the global zeta function depends on the Dedekind zeta function of K_2 (see below), the global zeta function has meromorphic continuation to $\mathbb{C}$.

7 Notes

Grunewald, Segal & Smith observed an interesting link with the Dedekind zeta function $\zeta_{K_2}(s)$ of the field K_2:

$$\zeta^{\triangleleft}_{H(\mathcal{O}_{K_2})}(s) = \zeta(s)\zeta(s-1)\zeta(s-2)\zeta(s-3)\zeta(5s-4)\zeta(5s-5)\zeta_{K_2}(3s-4)/\zeta_{K_2}(5s-4).$$

In particular, since the Dedekind zeta function has meromorphic continuation to $\mathbb{C}$, this tells us that the global zeta function also has meromorphic continuation to $\mathbb{C}$.