The zeta function of $H \times M_3$ counting ideals

1 Presentation

$H \times M_3$ has presentation

$$\langle t, z, u, v, x_1, x_2, x_3 \mid [t, u] = v, [z, x_1] = x_2, [z, x_2] = x_3 \rangle.$$ 

$H \times M_3$ has nilpotency class 3.

2 The local zeta function

The local zeta function was first calculated by Luke Woodward. It is

$$\zeta_{H \times M_3, p}(s) = \zeta_p(s)\zeta_p(s - 1)\zeta_p(s - 2)\zeta_p(s - 3)\zeta_p(3s - 4)^2\zeta_p(4s - 4)\zeta_p(5s - 5) \times \zeta_p(6s - 5)\zeta_p(7s - 6)\zeta_p(9s - 10)W(p, p^{-s})$$

where $W(X, Y)$ is

$$1 - 2X^4Y^5 + X^5Y^5 - X^4Y^6 + X^4Y^7 - 2X^5Y^7 + X^8Y^9 - 2X^9Y^9 + 3X^9Y^{11} - 2X^{10}Y^{11} + X^9Y^{12} + X^{10}Y^{13} + X^{13}Y^{14} + X^{14}Y^{15} - 2X^{13}Y^{16} + 3X^{14}Y^{16} - 2X^{14}Y^{18} + X^{15}Y^{18} - 2X^{18}Y^{20} + X^{19}Y^{20} - X^{19}Y^{21} + X^{18}Y^{22} - 2X^{19}Y^{22} + X^{23}Y^{27}.$$  

$\zeta_{H \times M_3}^{\sigma}(s)$ is uniform.

3 Functional equation

The local zeta function satisfies the functional equation

$$\zeta_{H \times M_3, p}^{\sigma}(s) \bigg|_{p \to p^{-1}} = -p^{21 - 14s} \zeta_{H \times M_3, p}^{\sigma}(s).$$

4 Abscissa of convergence and order of pole

The abscissa of convergence of $\zeta_{H \times M_3}^{\sigma}(s)$ is 4, with a simple pole at $s = 4$. 

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5 Ghost zeta function

The ghost zeta function is the product over all primes of

$$\zeta_p(s)\zeta_p(s-1)\zeta_p(s-2)\zeta_p(3s-4)^2\zeta_p(4s-4)\zeta_p(5s-5)\zeta_p(6s-5)$$

$$\times \zeta_p(7s-6)\zeta_p(9s-10)W_1(p,p^{-s})W_2(p,p^{-s})W_3(p,p^{-s})$$

where

$$W_1(X,Y) = 1 + X^5Y^5 - 2X^9Y^9,$$

$$W_2(X,Y) = -2 + X^{10}Y^{11},$$

$$W_3(X,Y) = 1 + X^4Y^7.$$

The ghost is unfriendly.

6 Natural boundary

$\zeta_{H \times M_3}^<(s)$ has a natural boundary at $\Re(s) = 1$, and is of type I.