The zeta function of $H \times \mathbb{Z}$ counting ideals

1 Presentation

$H \times \mathbb{Z}$ has presentation

$$\langle x, y, z, w \mid [x, y] = z \rangle.$$

$H \times \mathbb{Z}$ has nilpotency class 2.

2 The local zeta function

The local zeta function was first calculated by Grunewald, Segal & Smith. It is

$$\zeta_{H \times \mathbb{Z}, p}^{\triangleleft}(s) = \zeta_p(s)\zeta_p(s-1)\zeta_p(s-2)\zeta_p(3s-3).$$

$\zeta_{H \times \mathbb{Z}}^{\triangleleft}(s)$ is uniform.

3 Functional equation

The local zeta function satisfies the functional equation

$$\left.\zeta_{H \times \mathbb{Z}, p}^{\triangleleft}(s)\right|_{p \to p^{-1}} = p^{6-6s}\zeta_{H \times \mathbb{Z}, p}^{\triangleleft}(s).$$

4 Abscissa of convergence and order of pole

The abscissa of convergence of $\zeta_{H \times \mathbb{Z}}^{\triangleleft}(s)$ is 3, with a simple pole at $s = 3$.

5 Ghost zeta function

This zeta function is its own ghost.

6 Natural boundary

$\zeta_{H \times \mathbb{Z}}^{\triangleleft}(s)$ has meromorphic continuation to the whole of $\mathbb{C}$.