The zeta function of $H \times \mathbb{Z}$ counting all subrings

1 Presentation

$H \times \mathbb{Z}$ has presentation

$$\langle x, y, z, w \mid [x, y] = z \rangle \,.$$

$H \times \mathbb{Z}$ has nilpotency class 2.

2 The local zeta function

The local zeta function was first calculated by Luke Woodward. It is

$$\zeta_{H\times\mathbb{Z},p}(s) = \zeta_p(s)\zeta_p(s-1)\zeta_p(s-2)\zeta_p(2s-3)\zeta_p(2s-4)\zeta_p(3s-4)^{-1}.$$

$\zeta_{H\times\mathbb{Z}}(s)$ is uniform.

3 Functional equation

The local zeta function satisfies the functional equation

$$\zeta_{H\times\mathbb{Z},p}(s)\big|_{p\to p^{-1}} = p^{6-4s}\zeta_{H\times\mathbb{Z},p}(s).$$

4 Abscissa of convergence and order of pole

The abscissa of convergence of $\zeta_{H\times\mathbb{Z}}(s)$ is 3, with a simple pole at $s = 3$.

5 Ghost zeta function

This zeta function is its own ghost.

6 Natural boundary

$\zeta_{H\times\mathbb{Z}}(s)$ has meromorphic continuation to the whole of $\mathbb{C}$.