The zeta function of \( H \times \mathbb{Z} \) counting all subrings

1 Presentation

\( H \times \mathbb{Z} \) has presentation
\[
\langle x, y, z, w \mid [x, y] = z \rangle.
\]

\( H \times \mathbb{Z} \) has nilpotency class 2.

2 The local zeta function

The local zeta function was first calculated by Luke Woodward. It is
\[
\zeta_{H \times \mathbb{Z}, p}(s) = \zeta_p(s)\zeta_p(s-1)\zeta_p(2s-2)\zeta_p(2s-3)\zeta_p(2s-4)\zeta_p(3s-4)^{-1}.
\]
\( \zeta_{H \times \mathbb{Z}}(s) \) is uniform.

3 Functional equation

The local zeta function satisfies the functional equation
\[
\zeta_{H \times \mathbb{Z}, p}(s)|_{p \to p^{-1}} = p^{5-4s}\zeta_{H \times \mathbb{Z}, p}(s).
\]

4 Abscissa of convergence and order of pole

The abscissa of convergence of \( \zeta_{H \times \mathbb{Z}}(s) \) is 3, with a simple pole at \( s = 3 \).

5 Ghost zeta function

This zeta function is its own ghost.

6 Natural boundary

\( \zeta_{H \times \mathbb{Z}}(s) \) has meromorphic continuation to the whole of \( \mathbb{C} \).