The zeta function of $H \times \mathfrak{g}_{6,12}$ counting ideals

1 Presentation

$H \times \mathfrak{g}_{6,12}$ has presentation

$\langle t, u, x_1, x_2, x_3, x_4, v, x_5, x_6 \mid [t, u] = v, [x_1, x_3] = x_5, [x_1, x_5] = x_6, [x_2, x_4] = x_6 \rangle$.

$H \times \mathfrak{g}_{6,12}$ has nilpotency class 3.

2 The local zeta function

The local zeta function was first calculated by Luke Woodward. It is

$$\begin{aligned}\zeta^{\triangleleft}_{H\times\mathfrak{g}_{6,12},p}(s) = {} & \zeta_p(s)\zeta_p(s-1)\zeta_p(s-2)\zeta_p(s-3)\zeta_p(s-4)\zeta_p(s-5)\zeta_p(3s-6)^2 \\ & \times \zeta_p(5s-7)\zeta_p(6s-6)\zeta_p(7s-7)\zeta_p(8s-7)\zeta_p(9s-8) \\ & \times \zeta_p(11s-14)W(p, p^{-s})\end{aligned}$$

where $W(X, Y)$ is

$$\begin{aligned}&1 - X^6Y^5 - X^6Y^7 - X^6Y^8 + X^6Y^9 - 2X^7Y^9 + X^{12}Y^{11} - 2X^{13}Y^{11} \\ &+ 2X^{13}Y^{12} - X^{14}Y^{12} + 2X^{13}Y^{13} - X^{14}Y^{13} + X^{14}Y^{14} + 2X^{13}Y^{15} \\ &- X^{14}Y^{15} + X^{14}Y^{16} + X^{20}Y^{16} + X^{14}Y^{17} + X^{20}Y^{18} + X^{20}Y^{19} - 2X^{19}Y^{20} \\ &+ 2X^{21}Y^{20} - X^{20}Y^{21} - X^{20}Y^{22} - X^{26}Y^{23} - X^{20}Y^{24} - X^{26}Y^{24} + X^{26}Y^{25} \\ &- 2X^{27}Y^{25} - X^{26}Y^{26} + X^{26}Y^{27} - 2X^{27}Y^{27} + X^{26}Y^{28} - 2X^{27}Y^{28} \\ &+ 2X^{27}Y^{29} - X^{28}Y^{29} + 2X^{33}Y^{31} - X^{34}Y^{31} + X^{34}Y^{32} + X^{34}Y^{33} + X^{34}Y^{35} \\ &- X^{40}Y^{40}.\end{aligned}$$

$\zeta^{\triangleleft}_{H\times\mathfrak{g}_{6,12}}(s)$ is uniform.

3 Functional equation

The local zeta function satisfies the functional equation

$$\zeta^{\triangleleft}_{H\times\mathfrak{g}_{6,12},p}(s)\Big|_{p\to p^{-1}} = -p^{36-18s}\zeta^{\triangleleft}_{H\times\mathfrak{g}_{6,12},p}(s).$$

4 Abscissa of convergence and order of pole

The abscissa of convergence of $\zeta^{\triangleleft}_{H\times\mathfrak{g}_{6,12}}(s)$ is 6, with a simple pole at $s = 6$.

5 Ghost zeta function

The ghost zeta function is the product over all primes of

$$\zeta_p(s)\zeta_p(s-1)\zeta_p(s-2)\zeta_p(s-3)\zeta_p(s-4)\zeta_p(s-5)\zeta_p(3s-6)^2\zeta_p(5s-7)$$
$$\times \zeta_p(6s-6)\zeta_p(7s-7)\zeta_p(8s-7)\zeta_p(9s-8)\zeta_p(11s-14)W_1(p,p^{-s})W_2(p,p^{-s})$$
$$\times W_3(p,p^{-s})$$

where

$$W_1(X,Y) = 1 + X^{20}Y^{16},$$
$$W_2(X,Y) = 1 - X^{14}Y^{15},$$
$$W_3(X,Y) = -1 - X^{6}Y^{9}.$$

The ghost is friendly.

6 Natural boundary

$\zeta^{\triangleleft}_{H\times\mathfrak{g}_{6,12}}(s)$ has a natural boundary at $\Re(s) = 5/4$, and is of type III.