The zeta function of $L(E)$ counting ideals

1 Presentation

$L(E)$ has presentation

$$\left\langle \begin{matrix} x_1, x_2, x_3, x_4, x_5, \\ x_6, y_1, y_2, y_3 \end{matrix} \;\middle|\; \begin{matrix} [x_1, x_4] = y_3, [x_1, x_5] = y_1, [x_1, x_6] = x_2, [x_2, x_4] = y_2, \\ [x_2, x_6] = y_1, [x_3, x_4] = y_1, [x_3, x_5] = y_3 \end{matrix} \right\rangle.$$

$L(E)$ has nilpotency class 2.

2 The local zeta function

The local zeta functions for all but finitely many primes were first calculated by Christopher Voll. Let $|E(\mathbb{F}_p)|$ denote the number of points on the elliptic curve $Y^2Z = X^3 - XZ^2 \subseteq \mathbb{P}^2(\mathbb{F}_p)$. Then, for all but finitely many primes,

$$\begin{aligned} \zeta^{\triangleleft}_{L(E),p}(s) = {} & \zeta_p(s)\zeta_p(s-1)\zeta_p(s-2)\zeta_p(s-3)\zeta_p(s-4)\zeta_p(s-5)\zeta_p(s-6) \\ & \times \zeta_p(5s-7)\zeta_p(7s-8)\zeta_p(8s-14)\zeta_p(9s-18) \\ & \times (W_1(p, p^{-s}) + |E(\mathbb{F}_p)| W_2(p, p^{-s})) \end{aligned}$$

where

$$\begin{aligned} W_1(X,Y) &= (1 + X^6Y^7 + X^7Y^7 + X^{12}Y^8 + X^{13}Y^8 + X^{19}Y^{15})(1 - X^7Y^5) \\ W_2(X,Y) &= X^6Y^5(1 - Y^2)(1 + X^{13}Y^8) \end{aligned}$$

$\zeta^{\triangleleft}_{L(E)}(s)$ is non-uniform.

3 Functional equation

For all but finitely many primes,

$$\zeta^{\triangleleft}_{L(E),p}(s)|_{p\to p^{-1}} = -p^{36-15s}\zeta^{\triangleleft}_{L(E),p}(s).$$

4 Abscissa of convergence and order of pole

The abscissa of convergence and order of pole are unknown since the local zeta functions at finitely many primes are unknown.

5 Ghost zeta function

The ghost zeta function is unknown, since the local zeta functions vary with the prime.

6 Natural boundary

The natural boundary of $\zeta^{\triangleleft}_{L(E)}(s)$ is unknown.

7 Notes

Despite the fact that we now have a term counting the number of points on an elliptic curve mod p, the functional equation persists. Indeed, this term plays a key part, thanks to the functional equation of the Weil zeta function.