The zeta function of \( L(E) \) counting ideals

1 Presentation

\( L(E) \) has presentation

\[
\left\langle x_1, x_2, x_3, x_4, x_5, x_6, y_1, y_2, y_3 \right| \begin{align*}
[x_1, x_4] &= y_3, \quad [x_1, x_5] = y_1, \quad [x_1, x_6] = x_2, \quad [x_2, x_4] = y_2, \\
[x_2, x_6] &= y_1, \quad [x_3, x_4] = y_1, \quad [x_3, x_5] = y_3
\end{align*} \right\}.
\]

\( L(E) \) has nilpotency class 2.

2 The local zeta function

The local zeta functions for all but finitely many primes were first calculated by Christopher Voll. Let \( |E(F_p)| \) denote the number of points on the elliptic curve \( Y^2Z = X^3 - XZ^2 \subseteq \mathbb{P}^2(F_p) \). Then, for all but finitely many primes,

\[
\zeta_{L(E),p}(s) = \zeta_p(s) \frac{\zeta_p(s - 1) \zeta_p(s - 2) \zeta_p(s - 3) \zeta_p(s - 4) \zeta_p(s - 5) \zeta_p(s - 6)}{\zeta_p(5s - 7) \zeta_p(7s - 8) \zeta_p(8s - 14) \zeta_p(9s - 18)} \times (W_1(p, p^{-s}) + |E(F_p)|W_2(p, p^{-s}))
\]

where

\[
\begin{align*}
W_1(X,Y) &= (1 + X^6Y^7 + X^7Y^7 + X^{12}Y^8 + X^{13}Y^8 + X^{19}Y^{15})(1 - X^7Y^5) \\
W_2(X,Y) &= X^6Y^5(1 - Y^2)(1 + X^{13}Y^8)
\end{align*}
\]

\( \zeta_{L(E)}^\vartriangle(s) \) is non-uniform.

3 Functional equation

For all but finitely many primes,

\[
\zeta_{L(E),p}^\vartriangle(s) \big|_{p \to p^{-1}} = -p^{26-15s} \zeta_{L(E),p}^\vartriangle(s).
\]

4 Abscissa of convergence and order of pole

The abscissa of convergence and order of pole are unknown since the local zeta functions at finitely many primes are unknown.
5 Ghost zeta function

The ghost zeta function is unknown, since the local zeta functions vary with the prime.

6 Natural boundary

The natural boundary of $\zeta_{\text{L}(E)}(s)$ is unknown.

7 Notes

Despite the fact that we now have a term counting the number of points on an elliptic curve mod $p$, the functional equation persists. Indeed, this term plays a key part, thanks to the functional equation of the Weil zeta function.