The zeta function of L_{W} counting all subrings

1 Presentation

L_{W} has presentation

$$\langle z, w_1, w_2, x_1, x_2, y_1 \mid [z, w_1] = x_1, [z, w_2] = x_2, [z, x_1] = y_1 \rangle .$$

L_{W} has nilpotency class 3.

2 The local zeta function

The local zeta function was first calculated by Luke Woodward. It is

$$\begin{aligned}\zeta_{L_{\mathrm{W}},p}(s) &= \zeta_p(s)\zeta_p(s-1)\zeta_p(s-2)\zeta_p(2s-4)\zeta_p(2s-5)^2\zeta_p(3s-7)\zeta_p(3s-8)\\ &\quad\times \zeta_p(4s-10)\zeta_p(5s-12)W(p,p^{-s})\end{aligned}$$

where $W(X,Y)$ is

$$\begin{aligned}&1 + X^3Y^2 + X^4Y^2 - X^4Y^3 - X^5Y^3 + X^6Y^3 + X^7Y^3 - 2X^7Y^4 - 2X^8Y^4\\ &+ X^9Y^5 - 2X^{10}Y^5 - 3X^{11}Y^5 + X^{11}Y^6 + X^{12}Y^6 - 2X^{13}Y^6 - 3X^{14}Y^6\\ &+ X^{13}Y^7 + X^{14}Y^7 + 3X^{15}Y^7 - 2X^{16}Y^7 - X^{17}Y^7 + X^{16}Y^8 + X^{17}Y^8\\ &+ 2X^{18}Y^8 + 2X^{18}Y^9 + 2X^{21}Y^9 + 2X^{21}Y^{10} + X^{22}Y^{10} + X^{23}Y^{10} - X^{22}Y^{11}\\ &- 2X^{23}Y^{11} + 3X^{24}Y^{11} + X^{25}Y^{11} + X^{26}Y^{11} - 3X^{25}Y^{12} - 2X^{26}Y^{12}\\ &+ X^{27}Y^{12} + X^{28}Y^{12} - 3X^{28}Y^{13} - 2X^{29}Y^{13} + X^{30}Y^{13} - 2X^{31}Y^{14}\\ &- 2X^{32}Y^{14} + X^{32}Y^{15} + X^{33}Y^{15} - X^{34}Y^{15} - X^{35}Y^{15} + X^{35}Y^{16} + X^{36}Y^{16}\\ &+ X^{39}Y^{18}.\end{aligned}$$

$\zeta_{L_{\mathrm{W}}}(s)$ is uniform.

3 Functional equation

The local zeta function satisfies the functional equation

$$\zeta_{L_{\mathrm{W}},p}(s)\big|_{p\to p^{-1}} = p^{15-6s}\zeta_{L_{\mathrm{W}},p}(s).$$

4 Abscissa of convergence and order of pole

The abscissa of convergence of $\zeta_{L_{\mathrm{W}}}(s)$ is 3, with a quadruple pole at $s = 3$.

5 Ghost zeta function

The ghost zeta function is the product over all primes of

$$\zeta_p(s)\zeta_p(s-1)\zeta_p(s-2)\zeta_p(2s-4)\zeta_p(2s-5)^2\zeta_p(3s-7)\zeta_p(3s-8)\zeta_p(4s-10)$$
$$\times\,\zeta_p(5s-12)W_1(p,p^{-s})W_2(p,p^{-s})W_3(p,p^{-s})$$

where

$$\begin{aligned}
W_1(X,Y) &= 1 - X^{17}Y^7,\\
W_2(X,Y) &= -1 + X^9Y^4 - X^{18}Y^8,\\
W_3(X,Y) &= -1 + X^4Y^3.
\end{aligned}$$

The ghost is friendly.

6 Natural boundary

$\zeta_{L_{\mathrm{W}}}(s)$ has a natural boundary at $\Re(s) = 17/7$, and is of type III.