The zeta function of $H \times_{\mathbb{Z}} H$ counting ideals

1 Presentation

$H \times_{\mathbb{Z}} H$ has presentation

$$\langle x_1, x_2, x_3, x_4, y \mid [x_1, x_3] = y, [x_2, x_4] = y \rangle .$$

$H \times_{\mathbb{Z}} H$ has nilpotency class 2.

2 The local zeta function

The local zeta function was first calculated by Grunewald, Segal & Smith. It is

$$\zeta^{\triangleleft}_{H \times_{\mathbb{Z}} H, p}(s) = \zeta_p(s)\zeta_p(s-1)\zeta_p(s-2)\zeta_p(s-3)\zeta_p(5s-4).$$

$\zeta^{\triangleleft}_{H \times_{\mathbb{Z}} H}(s)$ is uniform.

3 Functional equation

The local zeta function satisfies the functional equation

$$\zeta^{\triangleleft}_{H \times_{\mathbb{Z}} H, p}(s)\Big|_{p \to p^{-1}} = -p^{10-9s}\zeta^{\triangleleft}_{H \times_{\mathbb{Z}} H, p}(s).$$

4 Abscissa of convergence and order of pole

The abscissa of convergence of $\zeta^{\triangleleft}_{H \times_{\mathbb{Z}} H}(s)$ is 4, with a simple pole at $s = 4$.

5 Ghost zeta function

This zeta function is its own ghost.

6 Natural boundary

$\zeta^{\triangleleft}_{H \times_{\mathbb{Z}} H}(s)$ has meromorphic continuation to the whole of $\mathbb{C}$.