The zeta function of $H \times_{\mathbb{Z}} H$ counting all subrings

1 Presentation

$H \times_{\mathbb{Z}} H$ has presentation

$$\langle x_1, x_2, x_3, x_4, y \mid [x_1, x_3] = y, [x_2, x_4] = y \rangle .$$

$H \times_{\mathbb{Z}} H$ has nilpotency class 2.

2 The local zeta function

The local zeta function was first calculated by Luke Woodward. It is

$$\zeta_{H \times_{\mathbb{Z}} H, p}(s) = \zeta_p(s)\zeta_p(s-1)\zeta_p(s-3)\zeta_p(3s-4)\zeta_p(3s-6)\zeta_p(3s-7)W(p, p^{-s})$$

where $W(X, Y)$ is

$$\begin{aligned} &1 + X^2Y + X^4Y^2 + X^5Y^3 + X^6Y^3 - X^5Y^4 - X^6Y^4 - X^7Y^5 - X^9Y^6 \\ &- X^{11}Y^7. \end{aligned}$$

$\zeta_{H \times_{\mathbb{Z}} H}(s)$ is uniform.

3 Functional equation

The local zeta function satisfies the functional equation

$$\left.\zeta_{H \times_{\mathbb{Z}} H, p}(s)\right|_{p \to p^{-1}} = -p^{10-5s}\zeta_{H \times_{\mathbb{Z}} H, p}(s).$$

4 Abscissa of convergence and order of pole

The abscissa of convergence of $\zeta_{H \times_{\mathbb{Z}} H}(s)$ is 4, with a simple pole at $s = 4$.

5 Ghost zeta function

The ghost zeta function is the product over all primes of

$$\zeta_p(s)\zeta_p(s-1)\zeta_p(s-3)\zeta_p(3s-4)\zeta_p(3s-6)\zeta_p(3s-7)W_1(p, p^{-s})W_2(p, p^{-s})$$

where

$$\begin{aligned} W_1(X, Y) &= 1 + X^2Y + X^4Y^2 + X^6Y^3, \\ W_2(X, Y) &= 1 - X^5Y^4. \end{aligned}$$

The ghost is friendly.

6 Natural boundary

$\zeta_{H\times_{\mathbb{Z}}H}(s)$ has a natural boundary at $\Re(s) = 2$, and is of type II.