The zeta function of $L_{\mathrm{W}2}$ counting ideals

1 Presentation

$L_{\mathrm{W}2}$ has presentation

$$\langle z, w_1, w_2, w_3, x_1, x_2, x_3, y \mid [z,w_1]=x_1, [z,w_2]=x_2, [z,w_3]=x_3, [z,x_1]=y \rangle .$$

$L_{\mathrm{W}2}$ has nilpotency class 3.

2 The local zeta function

The local zeta function was first calculated by Luke Woodward. It is

$$\begin{aligned}\zeta^{\triangleleft}_{L_{\mathrm{W}2},p}(s) &= \zeta_p(s)\zeta_p(s-1)\zeta_p(s-2)\zeta_p(s-3)\zeta_p(2s-3)\zeta_p(3s-6)\zeta_p(5s-7)\\ &\times \zeta_p(5s-10)\zeta_p(6s-10)\zeta_p(7s-12)\zeta_p(8s-12)\zeta_p(9s-17)\\ &\times \zeta_p(13s-23)W(p,p^{-s})\end{aligned}$$

where $W(X,Y)$ is

$$\begin{aligned}&1 - X^3Y^2 + X^4Y^3 + X^5Y^3 + X^6Y^4 - X^6Y^5 - X^7Y^5 + X^9Y^5 - X^{10}Y^7\\
&- X^{11}Y^8 - X^{12}Y^8 + X^{13}Y^8 - X^{12}Y^9 + X^{13}Y^9 - 2X^{14}Y^9 - X^{15}Y^9\\
&+ X^{14}Y^{10} - X^{16}Y^{10} - X^{17}Y^{10} + X^{15}Y^{11} - 2X^{16}Y^{11} - X^{18}Y^{11} + X^{20}Y^{11}\\
&+ X^{16}Y^{12} + X^{18}Y^{12} - X^{19}Y^{12} + X^{20}Y^{12} - X^{21}Y^{12} - X^{22}Y^{12} + X^{19}Y^{13}\\
&- X^{20}Y^{13} - 2X^{22}Y^{13} - X^{23}Y^{13} + 3X^{22}Y^{14} - 2X^{23}Y^{14} + X^{24}Y^{14}\\
&- X^{26}Y^{14} + X^{22}Y^{15} + X^{23}Y^{15} + X^{25}Y^{15} + X^{23}Y^{16} + X^{24}Y^{16} - 2X^{25}Y^{16}\\
&+ 2X^{26}Y^{16} - X^{27}Y^{16} - X^{25}Y^{17} + 2X^{26}Y^{17} + X^{27}Y^{17} - X^{28}Y^{17} - X^{30}Y^{17}\\
&- X^{26}Y^{18} + X^{27}Y^{18} + 2X^{28}Y^{18} + 2X^{29}Y^{18} - X^{30}Y^{18} + X^{31}Y^{18} - X^{29}Y^{19}\\
&+ 2X^{30}Y^{19} + X^{33}Y^{19} - X^{30}Y^{20} + X^{31}Y^{20} - X^{32}Y^{20} + 3X^{33}Y^{20} + X^{35}Y^{20}\\
&- 2X^{33}Y^{21} + 2X^{34}Y^{21} - X^{37}Y^{21} - X^{34}Y^{22} + 3X^{35}Y^{22} - 2X^{36}Y^{22}\\
&+ 3X^{37}Y^{22} + X^{39}Y^{22} - X^{34}Y^{23} - X^{35}Y^{23} - X^{37}Y^{23} + X^{38}Y^{23} + 3X^{40}Y^{23}\\
&- X^{41}Y^{23} - X^{38}Y^{24} + X^{39}Y^{24} - X^{40}Y^{24} + X^{41}Y^{24} - X^{42}Y^{24} - X^{38}Y^{25}\\
&- 2X^{39}Y^{25} - X^{40}Y^{25} - X^{41}Y^{25} + 2X^{42}Y^{25} - 2X^{43}Y^{25} + 2X^{44}Y^{25}\\
&+ X^{42}Y^{26} - X^{43}Y^{26} + 2X^{45}Y^{26} - 3X^{44}Y^{27} - 2X^{45}Y^{27} + 2X^{46}Y^{27}\\
&+ X^{48}Y^{27} - X^{45}Y^{28} - 2X^{47}Y^{28} - X^{48}Y^{28} + X^{49}Y^{28} + X^{44}Y^{29} - X^{45}Y^{29}\\
&- X^{46}Y^{29} - X^{49}Y^{29} - X^{50}Y^{29} - X^{49}Y^{30} + X^{50}Y^{30} - X^{51}Y^{30} - X^{52}Y^{30}\\
&+ X^{53}Y^{30} + X^{48}Y^{31} - X^{50}Y^{31} - 2X^{51}Y^{31} - 2X^{52}Y^{31} + 2X^{53}Y^{31}\end{aligned}$$

$$
\begin{aligned}
&- X^{54}Y^{31} + 2X^{51}Y^{32} - X^{53}Y^{32} - X^{55}Y^{32} - X^{56}Y^{32} + X^{52}Y^{33} - 3X^{56}Y^{33}\\
&- X^{54}Y^{34} + 2X^{55}Y^{34} + 2X^{56}Y^{34} + X^{56}Y^{35} - X^{58}Y^{35} - X^{60}Y^{35} + X^{57}Y^{36}\\
&+ X^{60}Y^{36} - X^{63}Y^{36} + X^{61}Y^{37} + X^{62}Y^{37} + X^{62}Y^{38} + X^{63}Y^{38} + X^{64}Y^{38}\\
&- X^{65}Y^{38} + X^{66}Y^{38} - X^{65}Y^{40} + X^{66}Y^{40} + X^{67}Y^{40} + X^{68}Y^{40} - X^{69}Y^{40}\\
&+ X^{69}Y^{41} - X^{69}Y^{42} + X^{72}Y^{42} + X^{71}Y^{43} - X^{72}Y^{43} - X^{73}Y^{45} - X^{74}Y^{45}\\
&+ X^{75}Y^{45} - X^{78}Y^{47}.
\end{aligned}
$$

$\zeta^{\triangleleft}_{L_{\mathrm{W}2}}(s)$ is uniform.

3 Functional equation

The local zeta function satisfies no functional equation.

4 Abscissa of convergence and order of pole

The abscissa of convergence of $\zeta^{\triangleleft}_{L_{\mathrm{W}2}}(s)$ is 4, with a simple pole at $s = 4$.

5 Ghost zeta function

The ghost zeta function is the product over all primes of

$$
\begin{aligned}
&\zeta_p(s)\zeta_p(s-1)\zeta_p(s-2)\zeta_p(s-3)\zeta_p(2s-3)\zeta_p(3s-6)\zeta_p(5s-7)\zeta_p(5s-10)\\
&\times \zeta_p(6s-10)\zeta_p(7s-12)\zeta_p(8s-12)\zeta_p(9s-17)\zeta_p(13s-23)W_1(p,p^{-s})\\
&\times W_2(p,p^{-s})W_3(p,p^{-s})W_4(p,p^{-s})W_5(p,p^{-s})
\end{aligned}
$$

where

$$
\begin{aligned}
W_1(X,Y) &= 1 - X^{26}Y^{14},\\
W_2(X,Y) &= -1 + X^{22}Y^{13},\\
W_3(X,Y) &= 1 + X^{5}Y^{3} - X^{15}Y^{9},\\
W_4(X,Y) &= -1 + X^{3}Y^{2} - X^{6}Y^{4} + X^{9}Y^{6},\\
W_5(X,Y) &= 1 - X^{6}Y^{5}.
\end{aligned}
$$

The ghost is unfriendly.

6 Natural boundary

$\zeta^{\triangleleft}_{L_{\mathrm{W}2}}(s)$ has a natural boundary at $\Re(s) = 13/7$, and is of type III.

7 Notes

L_{W2} was constructed to be similar to L_{W}. It is thus unsurprising that $\zeta^{\triangleleft}_{L_{\mathrm{W2}},p}(s)$ has no functional equation.