The zeta function of $M_3$ counting all subrings

1 **Presentation**

$M_3$ has presentation

$\langle z, x_1, x_2, x_3 \mid [z, x_1] = x_2, [z, x_2] = x_3 \rangle$.

$M_3$ has nilpotency class 3.

2 **The local zeta function**

The local zeta function was first calculated by Gareth Taylor. It is

$$\zeta_{M_3, p}(s) = \zeta_p(s)\zeta_p(s - 1)\zeta_p(2s - 3)\zeta_p(3s - 5)\zeta_p(4s - 6)W(p, p^{-s})$$

where $W(X, Y)$ is

$$1 + X^2Y^2 + X^3Y^2 - X^3Y^3 + X^4Y^3 - X^5Y^4 + X^6Y^4 - X^6Y^5 - X^7Y^5 - X^9Y^7.$$  

$\zeta_{M_3}(s)$ is uniform.

3 **Functional equation**

The local zeta function satisfies the functional equation

$$\zeta_{M_3, p}(s) \bigg|_{p \to p^{-1}} = p^{5 - 4s}\zeta_{M_3, p}(s).$$

4 **Abscissa of convergence and order of pole**

The abscissa of convergence of $\zeta_{M_3}(s)$ is 2, with a quadruple pole at $s = 2$.

5 **Ghost zeta function**

The ghost zeta function is the product over all primes of

$$\zeta_p(s)\zeta_p(s - 1)\zeta_p(2s - 3)\zeta_p(3s - 5)\zeta_p(4s - 6)W_1(p, p^{-s})W_2(p, p^{-s})$$

where

$$W_1(X, Y) = 1 + X^3Y^2 + X^6Y^4,$$

$$W_2(X, Y) = 1 - XY - X^3Y^3.$$  

The ghost is unfriendly.
6 Natural boundary

ζ_{M_5}(s) has a natural boundary at \Re(s) = 3/2, and is of type III.