The zeta function of $M_3 \times \mathbb{Z}$ counting ideals

1 Presentation

$M_3 \times \mathbb{Z}$ has presentation

$$\langle z, x_1, x_2, a, x_3 \mid [z, x_1] = x_2, [z, x_2] = x_3 \rangle .$$

$M_3 \times \mathbb{Z}$ has nilpotency class 3.

2 The local zeta function

The local zeta function was first calculated by Luke Woodward. It is

$$\zeta^{\triangleleft}_{M_3 \times \mathbb{Z},p}(s) = \zeta_p(s)\zeta_p(s-1)\zeta_p(s-2)\zeta_p(3s-3)\zeta_p(4s-3)\zeta_p(5s-4) \times \zeta_p(5s-3)^{-1}.$$

$\zeta^{\triangleleft}_{M_3 \times \mathbb{Z}}(s)$ is uniform.

3 Functional equation

The local zeta function satisfies the functional equation

$$\left.\zeta^{\triangleleft}_{M_3 \times \mathbb{Z},p}(s)\right|_{p \to p^{-1}} = -p^{10-10s}\zeta^{\triangleleft}_{M_3 \times \mathbb{Z},p}(s).$$

4 Abscissa of convergence and order of pole

The abscissa of convergence of $\zeta^{\triangleleft}_{M_3 \times \mathbb{Z}}(s)$ is 3, with a simple pole at $s = 3$.

5 Ghost zeta function

This zeta function is its own ghost.

6 Natural boundary

$\zeta^{\triangleleft}_{M_3 \times \mathbb{Z}}(s)$ has meromorphic continuation to the whole of $\mathbb{C}$.