The zeta function of \( M_3 \times \mathbb{Z} \) counting all subrings

1 Presentation

\( M_3 \times \mathbb{Z} \) has presentation

\[ \langle z, x_1, x_2, a, x_3 \mid [z, x_1] = x_2, [z, x_2] = x_3 \rangle. \]

\( M_3 \times \mathbb{Z} \) has nilpotency class 3.

2 The local zeta function

The local zeta function was first calculated by Luke Woodward. It is

\[ \zeta_{M_3 \times \mathbb{Z},p}(s) = \zeta_p(s)\zeta_p(s - 1)\zeta_p(s - 2)\zeta_p(2s - 4)\zeta_p(3s - 6)\zeta_p(4s - 8)W(p, p^{-s}) \]

where \( W(X,Y) \) is

\[ 1 - X^2Y + X^3Y^2 + X^4Y^2 - X^4Y^3 - X^5Y^3 - X^8Y^5 - X^9Y^6 \]
\[ + X^{10}Y^6 - X^{11}Y^7 + X^{13}Y^8. \]

\( \zeta_{M_3 \times \mathbb{Z}}(s) \) is uniform.

3 Functional equation

The local zeta function satisfies the functional equation

\[ \zeta_{M_3 \times \mathbb{Z},p}(s) |_{p \to p^{-1}} = -p^{10-5s}\zeta_{M_3 \times \mathbb{Z},p}(s). \]

4 Abscissa of convergence and order of pole

The abscissa of convergence of \( \zeta_{M_3 \times \mathbb{Z}}(s) \) is 3, with a simple pole at \( s = 3 \).

5 Ghost zeta function

The ghost zeta function is the product over all primes of

\[ \zeta_p(s)\zeta_p(s - 1)\zeta_p(s - 2)^2\zeta_p(2s - 4)\zeta_p(3s - 6)\zeta_p(4s - 8)W_1(p, p^{-s})W_2(p, p^{-s}) \]
\[ \times W_3(p, p^{-s}) \]
where

\[ W_1(X,Y) = 1 - X^2Y + X^4Y^2, \]
\[ W_2(X,Y) = 1 - X^5Y^3, \]
\[ W_3(X,Y) = -1 + X^4Y^3. \]

The ghost is friendly.

6 Natural boundary

\( \zeta_{M_3 \times \mathbb{Z}}(s) \) has a natural boundary at \( \Re(s) = 2 \), and is of type II.