The zeta function of $M_3 \times_{\mathbb{Z}} M_3$ counting ideals

1 Presentation

$M_3 \times_{\mathbb{Z}} M_3$ has presentation

$$\langle z_1, z_2, w_1, w_2, x_1, x_2, y \mid [z_1, w_1] = x_1, [z_2, w_2] = x_2, [z_1, x_1] = y, [z_2, x_2] = y \rangle .$$

$M_3 \times_{\mathbb{Z}} M_3$ has nilpotency class 3.

2 The local zeta function

The local zeta function was first calculated by Luke Woodward. It is

$$\begin{aligned}\zeta^{\triangleleft}_{M_3 \times_{\mathbb{Z}} M_3, p}(s) = {} & \zeta_p(s)\zeta_p(s-1)\zeta_p(s-2)\zeta_p(s-3)\zeta_p(3s-4)^2\zeta_p(5s-5)\zeta_p(7s-4) \\ & \times \zeta_p(8s-5)\zeta_p(9s-6)\zeta_p(12s-10)W(p, p^{-s})\end{aligned}$$

where $W(X, Y)$ is

$$\begin{aligned}& 1 - X^4Y^5 - 2X^4Y^8 + X^5Y^8 + X^4Y^9 - 2X^5Y^9 + X^8Y^{12} - 2X^9Y^{12} + 3X^9Y^{13} \\ & - 2X^{10}Y^{13} + X^{10}Y^{14} + X^9Y^{17} + X^{14}Y^{17} + X^{13}Y^{20} - 2X^{13}Y^{21} + 3X^{14}Y^{21} \\ & - 2X^{14}Y^{22} + X^{15}Y^{22} - 2X^{18}Y^{25} + X^{19}Y^{25} + X^{18}Y^{26} - 2X^{19}Y^{26} \\ & - X^{19}Y^{29} + X^{23}Y^{34}.\end{aligned}$$

$\zeta^{\triangleleft}_{M_3 \times_{\mathbb{Z}} M_3}(s)$ is uniform.

3 Functional equation

The local zeta function satisfies the functional equation

$$\zeta^{\triangleleft}_{M_3 \times_{\mathbb{Z}} M_3, p}(s)\Big|_{p \to p^{-1}} = -p^{21-17s}\zeta^{\triangleleft}_{M_3 \times_{\mathbb{Z}} M_3, p}(s).$$

4 Abscissa of convergence and order of pole

The abscissa of convergence of $\zeta^{\triangleleft}_{M_3 \times_{\mathbb{Z}} M_3}(s)$ is 4, with a simple pole at $s = 4$.

5 Ghost zeta function

The ghost zeta function is the product over all primes of

$$\begin{aligned}&\zeta_p(s)\zeta_p(s-1)\zeta_p(s-2)\zeta_p(s-3)\zeta_p(3s-4)^2\zeta_p(5s-5)\zeta_p(7s-4)\zeta_p(8s-5)\\&\times\zeta_p(9s-6)\zeta_p(12s-10)W_1(p,p^{-s})W_2(p,p^{-s})W_3(p,p^{-s})\end{aligned}$$

where

$$\begin{aligned}W_1(X,Y)&=1+X^{14}Y^{17},\\W_2(X,Y)&=1+X^{5}Y^{8},\\W_3(X,Y)&=1+X^{4}Y^{9}.\end{aligned}$$

The ghost is friendly.

6 Natural boundary

$\zeta^{\triangleleft}_{M_3\times_{\mathbb{Z}}M_3}(s)$ has a natural boundary at $\Re(s)=14/17$, and is of type III.

7 Notes

This ideal zeta function is identical to that of $\mathfrak{g}_{137B}$, though the Lie rings themselves are non-isomorphic.