The zeta function of $M_3 \times \mathbb{Z} M_3$ counting ideals

1 Presentation

$M_3 \times \mathbb{Z} M_3$ has presentation

$$\langle z_1, z_2, w_1, w_2, x_1, x_2, y \mid [z_1, w_1] = x_1, [z_2, w_2] = x_2, [z_1, x_1] = y, [z_2, x_2] = y \rangle.$$ 

$M_3 \times \mathbb{Z} M_3$ has nilpotency class 3.

2 The local zeta function

The local zeta function was first calculated by Luke Woodward. It is

$$\zeta_{M_3 \times \mathbb{Z} M_3, p} (s) = \zeta_p (s-1) \zeta_p (s-2) \zeta_p (3s-4) \zeta_p (7s-4) \cdot \zeta_p (8s-5) \zeta_p (9s-6) \zeta_p (12s-10) W(p, p^{-s})$$

where $W(X,Y)$ is

$$1 - X^4 Y^5 - 2X^4 Y^4 + X^5 Y^8 + X^4 Y^9 - 2X^5 Y^9 + X^8 Y^{12} - 2X^9 Y^{12} + 3X^9 Y^{13}$$

$$- 2X^{10} Y^{13} + X^{10} Y^{14} + X^9 Y^{17} + X^{14} Y^{17} + X^{13} Y^{20} - 2X^{13} Y^{21} + 3X^{14} Y^{21}$$

$$- 2X^{14} Y^{22} + X^{15} Y^{22} - 2X^{18} Y^{25} + X^{19} Y^{25} + X^{18} Y^{26} - 2X^{19} Y^{26}$$

$$- X^{19} Y^{29} + X^{23} Y^{34}.$$ 

$\zeta_{M_3 \times \mathbb{Z} M_3} (s)$ is uniform.

3 Functional equation

The local zeta function satisfies the functional equation

$$\zeta_{M_3 \times \mathbb{Z} M_3, p} (s) \Big|_{p \to p^{-1}} = -p^{21-17s} \zeta_{M_3 \times \mathbb{Z} M_3, p} (s).$$

4 Abscissa of convergence and order of pole

The abscissa of convergence of $\zeta_{M_3 \times \mathbb{Z} M_3} (s)$ is 4, with a simple pole at $s = 4$. 

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5 Ghost zeta function

The ghost zeta function is the product over all primes of

\[ \zeta_p(s) \zeta_p(s - 1) \zeta_p(s - 2) \zeta_p(s - 3) \zeta_p(3s - 4) \zeta_p(5s - 5) \zeta_p(7s - 4) \zeta_p(8s - 5) \times \zeta_p(9s - 6) \zeta_p(12s - 10) W_1(p, p^{-s}) W_2(p, p^{-s}) W_3(p, p^{-s}) \]

where

\[ W_1(X, Y) = 1 + X^{14} Y^{17}, \]
\[ W_2(X, Y) = 1 + X^5 Y^8, \]
\[ W_3(X, Y) = 1 + X^4 Y^9. \]

The ghost is friendly.

6 Natural boundary

\( \zeta_{M_3 \times Z}^\downarrow (s) \) has a natural boundary at \( \Re(s) = 14/17 \), and is of type III.

7 Notes

This ideal zeta function is identical to that of \( g_{137B} \), though the Lie rings themselves are non-isomorphic.