The zeta function of M_4 counting all subrings

1 Presentation

M_4 has presentation

$$\langle z, x_1, x_2, x_3, x_4 \mid [z, x_1] = x_2, [z, x_2] = x_3, [z, x_3] = x_4 \rangle .$$

M_4 has nilpotency class 4.

2 The local zeta function

The local zeta function was first calculated by Gareth Taylor. It is

$$\begin{aligned}\zeta_{M_4,p}(s) = {} & \zeta_p(s)\zeta_p(s-1)\zeta_p(2s-3)\zeta_p(2s-4)\zeta_p(3s-6)\zeta_p(4s-7)\zeta_p(4s-8) \\ & \times \zeta_p(7s-12)W(p,p^{-s})\end{aligned}$$

where $W(X,Y)$ is

$$\begin{aligned}& 1 + X^2Y^2 + X^3Y^2 - X^3Y^3 + X^4Y^3 + 2X^5Y^3 - 2X^5Y^4 + X^7Y^4 - 2X^7Y^5 \\ & - X^8Y^5 + X^9Y^5 - 2X^9Y^6 - 2X^{10}Y^6 - X^{11}Y^6 + X^{10}Y^7 - 2X^{12}Y^7 - X^{13}Y^7 \\ & + X^{13}Y^8 - X^{14}Y^8 - X^{16}Y^9 + X^{15}Y^{10} + X^{17}Y^{11} - X^{18}Y^{11} + X^{18}Y^{12} \\ & + 2X^{19}Y^{12} - X^{21}Y^{12} + X^{20}Y^{13} + 2X^{21}Y^{13} + 2X^{22}Y^{13} - X^{22}Y^{14} \\ & + X^{23}Y^{14} + 2X^{24}Y^{14} - X^{24}Y^{15} + 2X^{26}Y^{15} - 2X^{26}Y^{16} - X^{27}Y^{16} \\ & + X^{28}Y^{16} - X^{28}Y^{17} - X^{29}Y^{17} - X^{31}Y^{19}.\end{aligned}$$

$\zeta_{M_4}(s)$ is uniform.

3 Functional equation

The local zeta function satisfies the functional equation

$$\zeta_{M_4,p}(s)\big|_{p\to p^{-1}} = -p^{10-5s}\zeta_{M_4,p}(s).$$

4 Abscissa of convergence and order of pole

The abscissa of convergence of $\zeta_{M_4}(s)$ is $5/2$, with a simple pole at $s = 5/2$.

5 Ghost zeta function

The ghost zeta function is the product over all primes of

$$\zeta_p(s)\zeta_p(s-1)\zeta_p(2s-3)\zeta_p(2s-4)\zeta_p(3s-6)\zeta_p(4s-7)\zeta_p(4s-8)\zeta_p(7s-12)$$
$$\times W_1(p,p^{-s})W_2(p,p^{-s})W_3(p,p^{-s})$$

where

$$\begin{aligned} W_1(X,Y) &= 1 - X^{13}Y^7, \\ W_2(X,Y) &= -1 + X^{15}Y^9, \\ W_3(X,Y) &= 1 - XY - X^3Y^3. \end{aligned}$$

The ghost is unfriendly.

6 Natural boundary

$\zeta_{M_4}(s)$ has a natural boundary at $\Re(s) = 13/7$, and is of type III.

7 Notes

This calculation, a lengthy one by Gareth Taylor, is an example where the abscissa of convergence is not an integer. It is also the only calculation of a zeta function counting all subalgebras at class 4.