The zeta function of $M_4$

counting all subrings

1 Presentation

$M_4$ has presentation

$$\langle z, x_1, x_2, x_3, x_4 \mid [z, x_1] = x_2, [z, x_2] = x_3, [z, x_3] = x_4 \rangle.$$ 

$M_4$ has nilpotency class 4.

2 The local zeta function

The local zeta function was first calculated by Gareth Taylor. It is

$$\zeta_{M_4, p}(s) = \zeta_p(s - 1)\zeta_p(2s - 3)\zeta_p(2s - 4)\zeta_p(3s - 6)\zeta_p(4s - 7)\zeta_p(4s - 8) \times \zeta_p(7s - 12)W(p, p^{-s})$$

where $W(X, Y)$ is

$$1 + X^2Y^2 + X^3Y^2 - X^3Y^3 + X^4Y^3 + 2X^5Y^3 - 2X^5Y^4 + X^7Y^4 - 2X^7Y^5 - X^8Y^5 + X^9Y^5 - 2X^9Y^6 - 2X^{10}Y^6 - X^{11}Y^6 + X^{10}Y^7 - 2X^{12}Y^7 - X^{13}Y^7 + X^{13}Y^8 - X^{14}Y^8 - X^{16}Y^9 + X^{15}Y^{10} + X^{17}Y^{11} - X^{18}Y^{11} + X^{18}Y^{12} + 2X^{19}Y^{12} - X^{21}Y^{12} + X^{20}Y^{13} + 2X^{21}Y^{13} + 2X^{22}Y^{13} - X^{22}Y^{14} + X^{23}Y^{14} + 2X^{24}Y^{14} - X^{24}Y^{15} + 2X^{26}Y^{15} - 2X^{26}Y^{16} - X^{27}Y^{16} + X^{28}Y^{16} - X^{28}Y^{17} - X^{29}Y^{17} - X^{31}Y^{19}.$$

$\zeta_{M_4}(s)$ is uniform.

3 Functional equation

The local zeta function satisfies the functional equation

$$\zeta_{M_4, p}(s)|_{p \rightarrow p^{-1}} = -p^{10 - 5s}\zeta_{M_4, p}(s).$$

4 Abscissa of convergence and order of pole

The abscissa of convergence of $\zeta_{M_4}(s)$ is 5/2, with a simple pole at $s = 5/2$. 

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5 Ghost zeta function

The ghost zeta function is the product over all primes of

\[ \zeta_p(s) \zeta_p(s-1) \zeta_p(2s-3) \zeta_p(2s-4) \zeta_p(3s-6) \zeta_p(4s-7) \zeta_p(4s-8) \zeta_p(7s-12) \times W_1(p,p^{-s}) W_2(p,p^{-s}) W_3(p,p^{-s}) \]

where

\[ W_1(X,Y) = 1 - X^{13} Y^7, \]
\[ W_2(X,Y) = -1 + X^{15} Y^9, \]
\[ W_3(X,Y) = 1 - XY - X^3 Y^3. \]

The ghost is unfriendly.

6 Natural boundary

\( \zeta_M(s) \) has a natural boundary at \( \Re(s) = 13/7 \), and is of type III.

7 Notes

This calculation, a lengthy one by Gareth Taylor, is an example where the abscissa of convergence is not an integer. It is also the only calculation of a zeta function counting all subalgebras at class 4.