The zeta function of $M_4 \times \mathbb{Z}$ counting ideals

1 Presentation

$M_4 \times \mathbb{Z}$ has presentation

$$\langle z, x_1, x_2, a, x_3, x_4 \mid [z, x_1] = x_2, [z, x_2] = x_3, [z, x_3] = x_4 \rangle .$$

$M_4 \times \mathbb{Z}$ has nilpotency class 4.

2 The local zeta function

The local zeta function was first calculated by Luke Woodward. It is

$$\begin{aligned} \zeta^{\triangleleft}_{M_4\times\mathbb{Z},p}(s) &= \zeta_p(s)\zeta_p(s-1)\zeta_p(s-2)\zeta_p(3s-3)\zeta_p(5s-3)\zeta_p(7s-5)\zeta_p(8s-7) \\ &\times \zeta_p(9s-8)\zeta_p(11s-8)\zeta_p(12s-9)\zeta_p(6s-4)^{-1}W(p,p^{-s}) \end{aligned}$$

where $W(X,Y)$ is

$$\begin{aligned} &1 + X^3Y^4 - X^3Y^5 + X^4Y^5 - X^3Y^6 + 2X^4Y^6 - X^4Y^7 - X^7Y^9 + X^8Y^{10} \\ &- 2X^7Y^{11} - X^9Y^{13} - X^{11}Y^{13} + X^{10}Y^{14} - X^{11}Y^{14} - X^{11}Y^{15} - X^{12}Y^{15} \\ &+ X^{12}Y^{16} - X^{12}Y^{17} - X^{13}Y^{17} + 2X^{12}Y^{18} - X^{13}Y^{18} + X^{14}Y^{19} - 2X^{15}Y^{19} \\ &+ X^{14}Y^{20} + X^{15}Y^{20} - X^{15}Y^{21} + X^{15}Y^{22} + X^{16}Y^{22} + X^{16}Y^{23} - X^{17}Y^{23} \\ &+ X^{16}Y^{24} + X^{18}Y^{24} + 2X^{20}Y^{26} - X^{19}Y^{27} + X^{20}Y^{28} + X^{23}Y^{30} - 2X^{23}Y^{31} \\ &+ X^{24}Y^{31} - X^{23}Y^{32} + X^{24}Y^{32} - X^{24}Y^{33} - X^{27}Y^{37}. \end{aligned}$$

$\zeta^{\triangleleft}_{M_4\times\mathbb{Z}}(s)$ is uniform.

3 Functional equation

The local zeta function satisfies the functional equation

$$\zeta^{\triangleleft}_{M_4\times\mathbb{Z},p}(s)\Big|_{p\to p^{-1}} = p^{15-15s}\zeta^{\triangleleft}_{M_4\times\mathbb{Z},p}(s).$$

4 Abscissa of convergence and order of pole

The abscissa of convergence of $\zeta^{\triangleleft}_{M_4\times\mathbb{Z}}(s)$ is 3, with a simple pole at $s = 3$.

5 Ghost zeta function

The ghost zeta function is the product over all primes of

$$\begin{aligned}&\zeta_p(s)\zeta_p(s-1)\zeta_p(s-2)\zeta_p(3s-3)\zeta_p(5s-3)\zeta_p(7s-5)\zeta_p(8s-7)\zeta_p(9s-8)\\&\times \zeta_p(11s-8)\zeta_p(12s-9)W_1(p,p^{-s})W_2(p,p^{-s})W_3(p,p^{-s})W_4(p,p^{-s})\end{aligned}$$

where

$$\begin{aligned}W_1(X,Y) &= 1 - X^{11}Y^{13},\\W_2(X,Y) &= -1 + X^{13}Y^{18},\\W_3(X,Y) &= 1 - X^{4}Y^{6},\\W_4(X,Y) &= -1 + X^{3}Y^{6}.\end{aligned}$$

The ghost is friendly.

6 Natural boundary

$\zeta^{\triangleleft}_{M_4\times\mathbb{Z}}(s)$ has a natural boundary at $\Re(s) = 11/13$, and is of type III.