The zeta function of Q_5 counting all subrings

1 Presentation

Q_5 has presentation

$$\langle x_1, x_2, x_3, x_4, x_5 \mid [x_1, x_2] = x_3, [x_1, x_3] = x_5, [x_2, x_4] = x_5 \rangle .$$

Q_5 has nilpotency class 3.

2 The local zeta function

The local zeta function was first calculated by Luke Woodward. It is

$$\begin{aligned}\zeta_{Q_5,p}(s) = {} & \zeta_p(s)\zeta_p(s-1)\zeta_p(s-2)\zeta_p(2s-4)\zeta_p(3s-4)\zeta_p(3s-6)\zeta_p(6s-11) \\ & \times \zeta_p(6s-12)W(p, p^{-s})\end{aligned}$$

where $W(X, Y)$ is

$$\begin{aligned}&1 + X^3Y^2 - X^4Y^3 + X^5Y^3 - X^5Y^4 + X^7Y^4 + X^8Y^4 - 2X^7Y^5 - 2X^8Y^5 \\ &- X^9Y^5 + X^8Y^6 + X^9Y^6 + X^{10}Y^6 - X^{10}Y^7 - 2X^{11}Y^7 - 2X^{12}Y^7 + X^{11}Y^8 \\ &+ X^{12}Y^8 - X^{14}Y^8 - X^{15}Y^8 + X^{15}Y^{10} + X^{16}Y^{10} - X^{18}Y^{10} - X^{19}Y^{10} \\ &+ 2X^{18}Y^{11} + 2X^{19}Y^{11} + X^{20}Y^{11} - X^{20}Y^{12} - X^{21}Y^{12} - X^{22}Y^{12} + X^{21}Y^{13} \\ &+ 2X^{22}Y^{13} + 2X^{23}Y^{13} - X^{22}Y^{14} - X^{23}Y^{14} + X^{25}Y^{14} - X^{25}Y^{15} + X^{26}Y^{15} \\ &- X^{27}Y^{16} - X^{30}Y^{18}.\end{aligned}$$

$\zeta_{Q_5}(s)$ is uniform.

3 Functional equation

The local zeta function satisfies the functional equation

$$\zeta_{Q_5,p}(s)\big|_{p\to p^{-1}} = -p^{10-5s}\zeta_{Q_5,p}(s).$$

4 Abscissa of convergence and order of pole

The abscissa of convergence of $\zeta_{Q_5}(s)$ is 3, with a simple pole at $s = 3$.

5 Ghost zeta function

The ghost zeta function is the product over all primes of

$$\zeta_p(s)\zeta_p(s-1)\zeta_p(s-2)\zeta_p(2s-4)\zeta_p(3s-4)\zeta_p(3s-6)\zeta_p(6s-11)\zeta_p(6s-12) \\ \times W_1(p,p^{-s})W_2(p,p^{-s})W_3(p,p^{-s})W_4(p,p^{-s})$$

where

$$\begin{aligned} W_1(X,Y) &= 1 + X^8Y^4, \\ W_2(X,Y) &= 1 - X^{11}Y^6, \\ W_3(X,Y) &= -1 - X^3Y^2 + X^6Y^4, \\ W_4(X,Y) &= 1 - X^5Y^4. \end{aligned}$$

The ghost is unfriendly.

6 Natural boundary

$\zeta_{Q_5}(s)$ has a natural boundary at $\Re(s) = 2$, and is of type II.

7 Notes

This Lie ring comes from the only Lie algebra of dimension 5 not previously encountered by Luke Woodward. The subscript 5 comes from the dimension, but there isn't any significance in the letter Q.