The zeta function of T_4 counting all subrings

1 Presentation

T_4 has presentation

$$\langle x_1, x_2, x_3, x_4, y_1, y_2, y_3 \mid [x_1, x_2] = y_1, [x_2, x_3] = y_2, [x_3, x_4] = y_3 \rangle .$$

T_4 has nilpotency class 2.

2 The local zeta function

The local zeta function was first calculated by Luke Woodward. It is

$$\begin{aligned} \zeta_{T_4,p}(s) = {} & \zeta_p(s)\zeta_p(s-1)\zeta_p(s-2)\zeta_p(s-3)\zeta_p(2s-5)^2\zeta_p(2s-6)^2\zeta_p(3s-6) \\ & \times \zeta_p(3s-8)^2\zeta_p(3s-9)\zeta_p(4s-12)\zeta_p(5s-14)W(p,p^{-s}) \end{aligned}$$

where $W(X,Y)$ is

$$\begin{aligned}
& 1 + X^4Y^2 + X^5Y^2 - 2X^5Y^3 - 3X^6Y^3 + X^6Y^4 - X^8Y^4 - 2X^9Y^4 - 2X^{10}Y^4 \\
& + X^{11}Y^5 - 2X^{12}Y^5 - 4X^{13}Y^5 - X^{14}Y^5 + X^{11}Y^6 + X^{12}Y^6 + 4X^{13}Y^6 \\
& + 2X^{14}Y^6 + 2X^{15}Y^6 - 2X^{16}Y^6 - 2X^{17}Y^6 + X^{14}Y^7 + 3X^{15}Y^7 + 2X^{16}Y^7 \\
& + 4X^{17}Y^7 + 2X^{18}Y^7 - X^{19}Y^7 + X^{20}Y^7 - X^{15}Y^8 + 3X^{18}Y^8 + 6X^{19}Y^8 \\
& + 4X^{20}Y^8 + 3X^{21}Y^8 - X^{18}Y^9 - 4X^{19}Y^9 - 4X^{20}Y^9 - 2X^{21}Y^9 + 5X^{22}Y^9 \\
& + 2X^{23}Y^9 + 4X^{24}Y^9 + 2X^{25}Y^9 - X^{20}Y^{10} - 3X^{22}Y^{10} - 8X^{23}Y^{10} \\
& - 3X^{24}Y^{10} - X^{25}Y^{10} + 2X^{26}Y^{10} + 2X^{27}Y^{10} + 2X^{23}Y^{11} - 2X^{24}Y^{11} \\
& - 4X^{25}Y^{11} - 8X^{26}Y^{11} - 11X^{27}Y^{11} - 4X^{28}Y^{11} + X^{29}Y^{11} + 3X^{30}Y^{11} \\
& + X^{25}Y^{12} + 4X^{26}Y^{12} + 3X^{27}Y^{12} + 2X^{28}Y^{12} - 6X^{29}Y^{12} - 11X^{30}Y^{12} \\
& - 6X^{31}Y^{12} - 4X^{32}Y^{12} + X^{33}Y^{12} + 2X^{29}Y^{13} + 6X^{30}Y^{13} + 5X^{31}Y^{13} \\
& - 5X^{33}Y^{13} - 6X^{34}Y^{13} - 2X^{35}Y^{13} - X^{31}Y^{14} + 4X^{32}Y^{14} + 6X^{33}Y^{14} \\
& + 11X^{34}Y^{14} + 6X^{35}Y^{14} - 2X^{36}Y^{14} - 3X^{37}Y^{14} - 4X^{38}Y^{14} - X^{39}Y^{14} \\
& - 3X^{34}Y^{15} - X^{35}Y^{15} + 4X^{36}Y^{15} + 11X^{37}Y^{15} + 8X^{38}Y^{15} + 4X^{39}Y^{15} \\
& + 2X^{40}Y^{15} - 2X^{41}Y^{15} - 2X^{37}Y^{16} - 2X^{38}Y^{16} + X^{39}Y^{16} + 3X^{40}Y^{16} \\
& + 8X^{41}Y^{16} + 3X^{42}Y^{16} + X^{44}Y^{16} - 2X^{39}Y^{17} - 4X^{40}Y^{17} - 2X^{41}Y^{17} \\
& - 5X^{42}Y^{17} + 2X^{43}Y^{17} + 4X^{44}Y^{17} + 4X^{45}Y^{17} + X^{46}Y^{17} - 3X^{43}Y^{18} \\
& - 4X^{44}Y^{18} - 6X^{45}Y^{18} - 3X^{46}Y^{18} + X^{49}Y^{18} - X^{44}Y^{19} + X^{45}Y^{19} \\
& - 2X^{46}Y^{19} - 4X^{47}Y^{19} - 2X^{48}Y^{19} - 3X^{49}Y^{19} - X^{50}Y^{19} + 2X^{47}Y^{20}
\end{aligned}$$

$$+ 2X^{48}Y^{20} - 2X^{49}Y^{20} - 2X^{50}Y^{20} - 4X^{51}Y^{20} - X^{52}Y^{20} - X^{53}Y^{20}$$
$$+ X^{50}Y^{21} + 4X^{51}Y^{21} + 2X^{52}Y^{21} - X^{53}Y^{21} + 2X^{54}Y^{22} + 2X^{55}Y^{22}$$
$$+ X^{56}Y^{22} - X^{58}Y^{22} + 3X^{58}Y^{23} + 2X^{59}Y^{23} - X^{59}Y^{24} - X^{60}Y^{24} - X^{64}Y^{26}.$$

$\zeta_{T_4}(s)$ is uniform.

3 Functional equation

The local zeta function satisfies the functional equation

$$\zeta_{T_4,p}(s)|_{p\to p^{-1}} = -p^{21-7s}\zeta_{T_4,p}(s).$$

4 Abscissa of convergence and order of pole

The abscissa of convergence of $\zeta_{T_4}(s)$ is 4, with a simple pole at $s = 4$.

5 Ghost zeta function

The ghost zeta function is the product over all primes of

$$\zeta_p(s)\zeta_p(s-1)\zeta_p(s-2)\zeta_p(s-3)\zeta_p(2s-5)^2\zeta_p(2s-6)^2\zeta_p(3s-6)\zeta_p(3s-8)^2$$
$$\times \zeta_p(3s-9)\zeta_p(4s-12)\zeta_p(5s-14)W_1(p,p^{-s})W_2(p,p^{-s})W_3(p,p^{-s})$$
$$\times W_4(p,p^{-s})W_5(p,p^{-s})$$

where

$$W_1(X,Y) = 1 + X^{20}Y^7,$$
$$W_2(X,Y) = 1 - X^{19}Y^7,$$
$$W_3(X,Y) = -1 + X^5Y^2 + X^{10}Y^4,$$
$$W_4(X,Y) = 1 - X^9Y^4,$$
$$W_5(X,Y) = -1 - X^6Y^4.$$

The ghost is unfriendly.

6 Natural boundary

$\zeta_{T_4}(s)$ has a natural boundary at $\Re(s) = 20/7$, and is of type III.