The zeta function of $\mathfrak{g}_{6,4}$ counting all subrings

1 Presentation

$\mathfrak{g}_{6,4}$ has presentation

$$\langle x_1, x_2, x_3, x_4, x_5, x_6 \mid [x_1, x_2] = x_5, [x_1, x_3] = x_6, [x_2, x_4] = x_6 \rangle .$$

$\mathfrak{g}_{6,4}$ has nilpotency class 2.

2 The local zeta function

The local zeta function was first calculated by Luke Woodward. It is

$$\zeta_{\mathfrak{g}_{6,4},p}(s) = \zeta_p(s)\zeta_p(s-1)\zeta_p(s-2)\zeta_p(s-3)\zeta_p(2s-5)\zeta_p(3s-5)\zeta_p(3s-7) \times \zeta_p(3s-8)\zeta_p(4s-9)\zeta_p(4s-11)\zeta_p(5s-12)W(p,p^{-s})$$

where $W(X,Y)$ is

$$\begin{aligned}
&1 + X^{4}Y^{2} - X^{5}Y^{3} + X^{6}Y^{3} - X^{6}Y^{4} - X^{7}Y^{4} - X^{9}Y^{4} + X^{10}Y^{4} - X^{9}Y^{5}\\
&- 2X^{10}Y^{5} - 3X^{11}Y^{5} - 2X^{12}Y^{5} - X^{13}Y^{5} + X^{10}Y^{6} + X^{11}Y^{6} + 2X^{12}Y^{6}\\
&+ X^{13}Y^{6} + X^{14}Y^{6} - X^{15}Y^{6} - X^{13}Y^{7} - X^{14}Y^{7} - 2X^{15}Y^{7} - X^{16}Y^{7}\\
&- X^{17}Y^{7} + X^{14}Y^{8} + 2X^{15}Y^{8} + 3X^{16}Y^{8} + 3X^{17}Y^{8} + X^{18}Y^{8} - X^{19}Y^{8}\\
&+ X^{20}Y^{8} - X^{17}Y^{9} + X^{18}Y^{9} + 2X^{19}Y^{9} + 2X^{20}Y^{9} + 2X^{21}Y^{9} + X^{22}Y^{9}\\
&+ X^{22}Y^{10} + X^{23}Y^{10} + X^{24}Y^{10} - X^{21}Y^{11} - X^{22}Y^{11} + X^{26}Y^{11} + X^{27}Y^{11}\\
&- X^{24}Y^{12} - X^{25}Y^{12} - X^{26}Y^{12} - X^{26}Y^{13} - 2X^{27}Y^{13} - 2X^{28}Y^{13}\\
&- 2X^{29}Y^{13} - X^{30}Y^{13} + X^{31}Y^{13} - X^{28}Y^{14} + X^{29}Y^{14} - X^{30}Y^{14} - 3X^{31}Y^{14}\\
&- 3X^{32}Y^{14} - 2X^{33}Y^{14} - X^{34}Y^{14} + X^{31}Y^{15} + X^{32}Y^{15} + 2X^{33}Y^{15}\\
&+ X^{34}Y^{15} + X^{35}Y^{15} + X^{33}Y^{16} - X^{34}Y^{16} - X^{35}Y^{16} - 2X^{36}Y^{16} - X^{37}Y^{16}\\
&- X^{38}Y^{16} + X^{35}Y^{17} + 2X^{36}Y^{17} + 3X^{37}Y^{17} + 2X^{38}Y^{17} + X^{39}Y^{17}\\
&- X^{38}Y^{18} + X^{39}Y^{18} + X^{41}Y^{18} + X^{42}Y^{18} - X^{42}Y^{19} + X^{43}Y^{19} - X^{44}Y^{20}\\
&- X^{48}Y^{22}.
\end{aligned}$$

$\zeta_{\mathfrak{g}_{6,4}}(s)$ is uniform.

3 Functional equation

The local zeta function satisfies the functional equation

$$\zeta_{\mathfrak{g}_{6,4},p}(s)\big|_{p\to p^{-1}} = p^{15-6s}\zeta_{\mathfrak{g}_{6,4},p}(s).$$

4 Abscissa of convergence and order of pole

The abscissa of convergence of $\zeta_{\mathfrak{g}_{6,4}}(s)$ is 4, with a simple pole at $s = 4$.

5 Ghost zeta function

The ghost zeta function is the product over all primes of

$$\begin{aligned}&\zeta_p(s)\zeta_p(s-1)\zeta_p(s-2)\zeta_p(s-3)\zeta_p(2s-5)\zeta_p(3s-5)\zeta_p(3s-7)\zeta_p(3s-8)\\&\times\zeta_p(4s-9)\zeta_p(4s-11)\zeta_p(5s-12)W_1(p,p^{-s})W_2(p,p^{-s})W_3(p,p^{-s})\\&\times W_4(p,p^{-s})\end{aligned}$$

where

$$\begin{aligned}W_1(X,Y)&=1-X^{13}Y^5,\\W_2(X,Y)&=-1+X^7Y^3+X^{14}Y^6-X^{21}Y^9,\\W_3(X,Y)&=-1-X^4Y^2+X^8Y^4,\\W_4(X,Y)&=1-X^6Y^4.\end{aligned}$$

The ghost is unfriendly.

6 Natural boundary

$\zeta_{\mathfrak{g}_{6,4}}(s)$ has a natural boundary at $\Re(s) = 13/5$, and is of type III.

7 Notes

This zeta function is a lot more complicated than the one only counting ideals!

The Lie ring is sometimes written as $(F_{2,3}/\langle z\rangle)\cdot\mathbb{Z}$.