The zeta function of $\mathfrak{g}_{6,8}$ counting ideals

1 Presentation

$\mathfrak{g}_{6,8}$ has presentation

$$\langle x_1, x_2, x_3, x_4, x_5, x_6 \mid [x_1, x_2] = x_3 + x_5, [x_1, x_3] = x_4, [x_2, x_5] = x_6 \rangle .$$

$\mathfrak{g}_{6,8}$ has nilpotency class 3.

2 The local zeta function

The local zeta function was first calculated by Luke Woodward. It is

$$\begin{aligned}\zeta^{\triangleleft}_{\mathfrak{g}_{6,8},p}(s) = {} & \zeta_p(s)\zeta_p(s-1)\zeta_p(s-2)\zeta_p(3s-3)\zeta_p(4s-3)\zeta_p(5s-5)\zeta_p(6s-6) \\ & \times \zeta_p(7s-7)\zeta_p(8s-8)W(p, p^{-s})\end{aligned}$$

where $W(X, Y)$ is

$$\begin{aligned} & 1 + X^3Y^4 - 2X^3Y^5 + X^4Y^5 - 2X^6Y^7 + X^7Y^7 + X^6Y^8 - 2X^7Y^8 - X^7Y^9 \\ & - X^{10}Y^{11} + X^9Y^{12} - X^{11}Y^{12} + X^{10}Y^{13} + X^{13}Y^{15} + 2X^{13}Y^{16} - X^{14}Y^{16} \\ & - X^{13}Y^{17} + 2X^{14}Y^{17} - X^{16}Y^{19} + 2X^{17}Y^{19} - X^{17}Y^{20} - X^{20}Y^{24}. \end{aligned}$$

$\zeta^{\triangleleft}_{\mathfrak{g}_{6,8}}(s)$ is uniform.

3 Functional equation

The local zeta function satisfies the functional equation

$$\zeta^{\triangleleft}_{\mathfrak{g}_{6,8},p}(s)\Big|_{p \to p^{-1}} = p^{15-12s} \zeta^{\triangleleft}_{\mathfrak{g}_{6,8},p}(s).$$

4 Abscissa of convergence and order of pole

The abscissa of convergence of $\zeta^{\triangleleft}_{\mathfrak{g}_{6,8}}(s)$ is 3, with a simple pole at $s = 3$.

5 Ghost zeta function

The ghost zeta function is the product over all primes of

$$\begin{aligned} & \zeta_p(s)\zeta_p(s-1)\zeta_p(s-2)\zeta_p(3s-3)\zeta_p(4s-3)\zeta_p(5s-5)\zeta_p(6s-6)\zeta_p(7s-7) \\ & \times \zeta_p(8s-8)W_1(p, p^{-s})W_2(p, p^{-s})W_3(p, p^{-s})\end{aligned}$$

where

$$
\begin{aligned}
W_1(X,Y) &= 1 + X^7Y^7,\\
W_2(X,Y) &= 1 + 2X^{10}Y^{12},\\
W_3(X,Y) &= 2 - X^3Y^5.
\end{aligned}
$$

The ghost is unfriendly.

6 Natural boundary

$\zeta^{\triangleleft}_{\mathfrak{g}_{6,8}}(s)$ has a natural boundary at $\Re(s) = 1$, and is of type II.