The zeta function of $\mathfrak{g}_{6,9}$ counting ideals

1 Presentation

$\mathfrak{g}_{6,9}$ has presentation

$$\langle x_1, x_2, x_3, x_4, x_5, x_6 \mid [x_1, x_2] = x_3, [x_1, x_3] = x_4, [x_1, x_5] = x_6, [x_2, x_3] = x_6 \rangle .$$

$\mathfrak{g}_{6,9}$ has nilpotency class 3.

2 The local zeta function

The local zeta function was first calculated by Luke Woodward. It is

$$\zeta^{\triangleleft}_{\mathfrak{g}_{6,9},p}(s) = \zeta_p(s)\zeta_p(s-1)^2\zeta_p(s-2)\zeta_p(5s-5)\zeta_p(6s-6)\zeta_p(8s-7)\zeta_p(8s-8) \times \zeta_p(14s-15)\zeta_p(2s-2)^{-1}W(p, p^{-s})$$

where $W(X,Y)$ is

$$1 - XY + X^2Y^2 + X^3Y^4 - X^3Y^5 + X^4Y^6 + X^6Y^6 - X^5Y^7 - X^7Y^9 + X^9Y^9 + X^8Y^{10} - 2X^9Y^{11} - X^{10}Y^{11} + X^{11}Y^{11} + X^{10}Y^{12} - 2X^{11}Y^{13} + X^{13}Y^{13} + X^{12}Y^{14} - X^{13}Y^{14} - X^{14}Y^{14} - 2X^{13}Y^{15} + X^{14}Y^{15} + X^{15}Y^{15} + X^{13}Y^{16} + X^{14}Y^{16} - 2X^{15}Y^{16} - X^{14}Y^{17} - X^{15}Y^{17} + X^{16}Y^{17} + X^{15}Y^{18} - 2X^{17}Y^{18} + X^{18}Y^{19} + X^{17}Y^{20} - X^{18}Y^{20} - 2X^{19}Y^{20} + X^{20}Y^{21} + X^{19}Y^{22} - X^{21}Y^{22} - X^{23}Y^{24} + X^{22}Y^{25} + X^{24}Y^{25} - X^{25}Y^{26} + X^{25}Y^{27} + X^{26}Y^{29} - X^{27}Y^{30} + X^{28}Y^{31}.$$

$\zeta^{\triangleleft}_{\mathfrak{g}_{6,9}}(s)$ is uniform.

3 Functional equation

The local zeta function satisfies the functional equation

$$\zeta^{\triangleleft}_{\mathfrak{g}_{6,9},p}(s)\Big|_{p\to p^{-1}} = p^{15-12s}\zeta^{\triangleleft}_{\mathfrak{g}_{6,9},p}(s).$$

4 Abscissa of convergence and order of pole

The abscissa of convergence of $\zeta^{\triangleleft}_{\mathfrak{g}_{6,9}}(s)$ is 3, with a simple pole at $s = 3$.

5 Ghost zeta function

The ghost zeta function is the product over all primes of

$$\zeta_p(s)\zeta_p(s-1)\zeta_p(s-2)\zeta_p(5s-5)\zeta_p(6s-6)\zeta_p(8s-7)\zeta_p(8s-8)\zeta_p(14s-15) \\ \times W_1(p,p^{-s})W_2(p,p^{-s})W_3(p,p^{-s})$$

where

$$\begin{aligned} W_1(X,Y) &= 1 + X^3Y^3 + X^6Y^6 + X^7Y^7 + X^9Y^9 + X^{10}Y^{10} + X^{11}Y^{11} \\ &\quad + X^{12}Y^{12} + X^{13}Y^{13} + X^{16}Y^{16}, \\ W_2(X,Y) &= 1 - X^{10}Y^{11}, \\ W_3(X,Y) &= -1 + X^3Y^5. \end{aligned}$$

The ghost is unfriendly.

6 Natural boundary

$\zeta^{\triangleleft}_{\mathfrak{g}_{6,9}}(s)$ has a natural boundary at $\Re(s) = 1$, and is of type I.