The zeta function of p3 counting all subgroups

1 Presentation

p3 has presentation

$$\left\langle x, y, r \mid [x,y], r^3, x^r = x^{-1}y, y^r = x^{-1} \right\rangle.$$

2 The zeta function itself

The zeta function was calculated by du Sautoy, McDermott and Smith. It is

$$\zeta_{\mathbf{p3}}(s) = \zeta(s-1)L(s-1,\chi_3) + 3^{-s}\zeta(s)\zeta(s-1),$$

where χ_3 is defined by

$$\chi_3(a) = \begin{cases} 1 & \text{if } a \equiv 1(\mod 3)) \\ -1 & \text{if } a \equiv -1(\mod 3)) \\ 0 & \text{otherwise} \end{cases},$$

and $L(s,\chi_3)$ is the Dirichlet L-function of χ_3,

$$L(s,\chi_3) = \sum_{n=1}^{\infty} \chi_3(n)n^{-s}.$$

3 Abscissa of convergence and order of pole

The abscissa of convergence of $\zeta_{\mathbf{p3}}(s)$ is 2, with a simple pole at $s = 2$. Since this group is a finite extension of a free abelian group, its zeta function has meromorphic continuation to $\mathbb{C}$.