# The zeta function of p4gm counting all subgroups

## 1 Presentation

**p4gm** has presentation

$$\left\langle x, y, r, t \mid [x, y], r^4, t^2, y^r = x^{-1}, x^r = y, x^t = y, r^t = r^{-1}x^{-1} \right\rangle .$$

## 2 The zeta function itself

The zeta function was calculated by du Sautoy, McDermott and Smith. It is

$$\begin{aligned}\zeta_{\mathbf{p4gm}}(s) = &(1 - 4 \cdot 4^{-s})\zeta(2s-2) + (4 \cdot 4^{-s} - 3 \cdot 8^{-s} + 8 \cdot 16^{-s})\zeta(s)\zeta(s-1) \\ &+ (2 \cdot 2^{-s} - 4 \cdot 4^{-s} + 12 \cdot 8^{-s})\zeta(s-1)^2 + 4^{-s}\zeta(s-1)\zeta(s-2) \\ &+ 2^{-s}\zeta(s-1)L(s-1, \chi_4),\end{aligned}$$

where $\chi_4$ is defined by

$$\chi_4(a) = \begin{cases} 1 & \text{if } a \equiv 1( \pmod 4) \\ -1 & \text{if } a \equiv -1( \pmod 4) \\ 0 & \text{otherwise} \end{cases},$$

and $L(s, \chi_4)$ is the Dirichlet $L$-function of $\chi_4$,

$$L(s, \chi_4) = \sum_{n=1}^{\infty} \chi_4(n) n^{-s}.$$

## 3 Abscissa of convergence and order of pole

The abscissa of convergence of $\zeta_{\mathbf{p4gm}}(s)$ is 3, with a simple pole at $s = 3$. Since this group is a finite extension of a free abelian group, its zeta function has meromorphic continuation to $\mathbb{C}$.