# The zeta function of p4mm counting all subgroups

## 1 Presentation

**p4mm** has presentation

$$\left\langle x, y, r, m \mid [x, y], r^4, m^2, y^r = x^{-1}, x^r = y, x^m = y, r^m = r^{-1} \right\rangle .$$

## 2 The zeta function itself

The zeta function was calculated by du Sautoy, McDermott and Smith. It is

$$\begin{aligned}\zeta_{\mathbf{p4mm}}(s) &= (1 + 4 \cdot 2^{-s} + 4 \cdot 4^{-s})\zeta(2s - 2) \\ &\quad + (4 \cdot 4^{-s} + 5 \cdot 8^{-s} + 8 \cdot 16^{-s})\zeta(s)\zeta(s - 1) \\ &\quad + (2 \cdot 2^{-s} + 8 \cdot 4^{-s} + 4 \cdot 8^{-s})\zeta(s - 1)^2 + 4^{-s}\zeta(s - 1)\zeta(s - 2) \\ &\quad + 2^{-s}\zeta(s - 1)L(s - 1, \chi_4),\end{aligned}$$

where $\chi_4$ is defined by

$$\chi_4(a) = \begin{cases} 1 & \text{if } a \equiv 1( \pmod 4) \\ -1 & \text{if } a \equiv -1( \pmod 4) \\ 0 & \text{otherwise} \end{cases},$$

and $L(s, \chi_4)$ is the Dirichlet $L$-function of $\chi_4$,

$$L(s, \chi_4) = \sum_{n=1}^{\infty} \chi_4(n)n^{-s}.$$

## 3 Abscissa of convergence and order of pole

The abscissa of convergence of $\zeta_{\mathbf{p4mm}}(s)$ is 3, with a simple pole at $s = 3$. Since this group is a finite extension of a free abelian group, its zeta function has meromorphic continuation to $\mathbb{C}$.