The zeta function of p6mm counting all subgroups

1 Presentation

p6mm has presentation

$$\left\langle x, y, r, m \;\middle|\; \begin{array}{c} [x,y], r^6, m^2, y^r = x^{-1}y, x^r = y, x^r = y, \\ x^m = x^{-1}, y^m = x^{-1}y, r^m = r^{-1}y \end{array} \right\rangle.$$

2 The zeta function itself

The zeta function was calculated by du Sautoy, McDermott and Smith. It is

$$\begin{aligned}
\zeta_{\mathbf{p6mm}}(s) = {} & (1 + 2 \cdot 2^{-s} + 3 \cdot 3^{-s} + 10 \cdot 6^{-s})\zeta(2s-2) \\
& + (2^{-s} + 4^{-s})\zeta(s-1)L(s-1,\chi_3) + (3 \cdot 3^{-s} + 24 \cdot 12^{-s})\zeta(s-1)^2 \\
& + 6^{-s}\zeta(s-1)\zeta(s-2) \\
& + (6 \cdot 6^{-s} - 5 \cdot 12^{-s} + 24 \cdot 24^{-s})\zeta(s)\zeta(s-1),
\end{aligned}$$

where χ_3 is defined by

$$\chi_3(a) = \begin{cases} 1 & \text{if } a \equiv 1(\pmod 3) \\ -1 & \text{if } a \equiv -1(\pmod 3) \\ 0 & \text{otherwise} \end{cases},$$

and $L(s, \chi_3)$ is the Dirichlet L-function of χ_3,

$$L(s, \chi_3) = \sum_{n=1}^{\infty} \chi_3(n) n^{-s}.$$

3 Abscissa of convergence and order of pole

The abscissa of convergence of $\zeta_{\mathbf{p6mm}}(s)$ is 3, with a simple pole at $s = 3$. Since this group is a finite extension of a free abelian group, its zeta function has meromorphic continuation to $\mathbb{C}$.