The zeta function of p6mm counting normal subgroups

1   Presentation

p6mm has presentation

$$\left\langle x, y, r, m \ \middle| \ \begin{array}{c} [x,y], r^6, m^2, y^r = x^{-1}y, x^r = y, x^r = y, \\ x^m = x^{-1}, y^m = x^{-1}y, r^m = r^{-1}y \end{array} \right\rangle.$$

2   The zeta function itself

The zeta function was calculated by du Sautoy, McDermott and Smith. It is

$$\zeta_{\mathbf{p6mm}}^{\triangleleft}(s) = 1 + 3 \cdot 2^{-s} + 4^{-s} + 2 \cdot 6^{-s} + 12^{-s} + 24^{-s} + (12^{-s} + 36^{-s})\zeta(2s).$$

3   Abscissa of convergence and order of pole

The abscissa of convergence of $\zeta_{\mathbf{p6mm}}^{\triangleleft}(s)$ is $1/2$, with a simple pole at $s = 1/2$. Since this group is a finite extension of a free abelian group, its zeta function has meromorphic continuation to $\mathbb{C}$.