The zeta function of $\mathfrak{sl}_2(\mathbb{Z})$ counting all subrings

1 Presentation

$\mathfrak{sl}_2(\mathbb{Z})$ has presentation

$$\langle f, e, h \mid [h, e] = 2e, [h, f] = -2f, [e, f] = h \rangle .$$

$\mathfrak{sl}_2(\mathbb{Z})$ is insoluble.

2 The local zeta function

The local zeta functions were calculated by Marcus du Sautoy using calculations by Ishai Ilani ($p \neq 2$), Marcus du Sautoy and Gareth Taylor (all p) and Juliette White (all p). For $p \neq 2$,

$$\zeta_{\mathfrak{sl}_2(\mathbb{Z}),p}(s) = \zeta_p(s)\zeta_p(s-1)\zeta_p(2s-1)\zeta_p(2s-2)\zeta_p(3s-1)^{-1}.$$

For $p = 2$,

$$\zeta_{\mathfrak{sl}_2(\mathbb{Z}),2}(s) = \zeta_2(s)\zeta_2(s-1)\zeta_2(2s-1)\zeta_2(2s-2)(1 + 6.2^{-2s} - 8.2^{-3s}).$$

3 Functional equation

For $p \neq 2$, the local zeta function satisfies the functional equation

$$\zeta_{\mathfrak{sl}_2(\mathbb{Z}),p}(s)|_{p \to p^{-1}} = -p^{3-3s}\zeta_{\mathfrak{sl}_2(\mathbb{Z}),p}(s).$$

$\zeta_{\mathfrak{sl}_2(\mathbb{Z}),2}(s)$ satisfies no functional equation.

4 Abscissa of convergence and order of pole

The abscissa of convergence of $\zeta_{\mathfrak{sl}_2(\mathbb{Z})}(s)$ is 2, with a simple pole at $s = 2$.

5 Ghost zeta function

The ghost zeta function is unknown since the zeta functions vary with the prime.

6 Natural boundary

$\zeta_{\mathfrak{sl}_2(\mathbb{Z})}(s)$ has meromorphic continuation to the whole of $\mathbb{C}$.