The zeta function of $\text{tr}_4(\mathbb{Z})$ counting ideals

1 Introduction

$\text{tr}_4(\mathbb{Z})$ is the Lie ring of upper-triangular $4 \times 4$ matrices over $\mathbb{Z}$.

2 The local zeta function

The local zeta function was first calculated by Luke Woodward. It is

$$\zeta_{\text{tr}_4(\mathbb{Z}),p}(s) = \zeta_p(s)^2 \zeta_p(s-1) \zeta_p(s-2) \zeta_p(s-3) \zeta_p(2s)^2 \zeta_p(5s) \zeta_p(8s) \zeta_p(9s) \times W(p, p^{-s})$$

where $W(X, Y)$ is


$\zeta_{\text{tr}_4(\mathbb{Z})}(s)$ is uniform.

3 Functional equation

The local zeta function satisfies the functional equation

$$\zeta_{\text{tr}_4(\mathbb{Z}),p}(s) \bigg|_{p \rightarrow p^{-1}} = p^{6 - 27s} \zeta_{\text{tr}_4(\mathbb{Z}),p}^{-1}(s).$$

4 Abscissa of convergence and order of pole

The abscissa of convergence of $\zeta_{\text{tr}_4(\mathbb{Z})}(s)$ is $4$, with a simple pole at $s = 4$.

5 Ghost zeta function

This zeta function is its own ghost.

6 Natural boundary

$\zeta_{\text{tr}_4(\mathbb{Z})}^{-1}(s)$ has meromorphic continuation to $\mathbb{C}$. 

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