# The zeta function of $\mathfrak{tr}_6(\mathbb{Z})$ counting ideals

## 1 Introduction

$\mathfrak{tr}_6(\mathbb{Z})$ is the Lie ring of upper-triangular $6 \times 6$ matrices over $\mathbb{Z}$.

## 2 The local zeta function

The local zeta function was first calculated by Luke Woodward. It is

$$\begin{aligned}\zeta^{\triangleleft}_{\mathfrak{tr}_6(\mathbb{Z}),p}(s) = {} & \zeta_p(s)\zeta_p(s-1)\zeta_p(s-2)\zeta_p(s-3)\zeta_p(s-4)\zeta_p(s-5)\zeta_p(2s)^3\zeta_p(5s)^2 \\ & \times \zeta_p(8s)\zeta_p(9s)\zeta_p(11s)\zeta_p(12s)\zeta_p(13s)\zeta_p(14s)\zeta_p(15s)\zeta_p(16s) \\ & \times \zeta_p(17s)\zeta_p(18s)\zeta_p(19s)\zeta_p(20s)W(p,p^{-s})\end{aligned}$$

where $W(X,Y)$ is

$$\begin{aligned} & 1 + 2Y^{2} + 3Y^{4} + 2Y^{5} + 4Y^{6} + 4Y^{7} + 7Y^{8} + 8Y^{9} + 10Y^{10} + 13Y^{11} + 16Y^{12} \\ & + 19Y^{13} + 24Y^{14} + 27Y^{15} + 34Y^{16} + 37Y^{17} + 44Y^{18} + 48Y^{19} + 56Y^{20} \\ & + 59Y^{21} + 70Y^{22} + 72Y^{23} + 81Y^{24} + 83Y^{25} + 90Y^{26} + 91Y^{27} + 95Y^{28} \\ & + 93Y^{29} + 99Y^{30} + 91Y^{31} + 92Y^{32} + 82Y^{33} + 80Y^{34} + 63Y^{35} + 62Y^{36} \\ & + 38Y^{37} + 34Y^{38} + 9Y^{39} - 27Y^{41} - 38Y^{42} - 68Y^{43} - 75Y^{44} - 105Y^{45} \\ & - 115Y^{46} - 139Y^{47} - 146Y^{48} - 173Y^{49} - 171Y^{50} - 195Y^{51} - 188Y^{52} \\ & - 206Y^{53} - 194Y^{54} - 206Y^{55} - 188Y^{56} - 195Y^{57} - 171Y^{58} - 173Y^{59} \\ & - 146Y^{60} - 139Y^{61} - 115Y^{62} - 105Y^{63} - 75Y^{64} - 68Y^{65} - 38Y^{66} - 27Y^{67} \\ & + 9Y^{69} + 34Y^{70} + 38Y^{71} + 62Y^{72} + 63Y^{73} + 80Y^{74} + 82Y^{75} + 92Y^{76} \\ & + 91Y^{77} + 99Y^{78} + 93Y^{79} + 95Y^{80} + 91Y^{81} + 90Y^{82} + 83Y^{83} + 81Y^{84} \\ & + 72Y^{85} + 70Y^{86} + 59Y^{87} + 56Y^{88} + 48Y^{89} + 44Y^{90} + 37Y^{91} + 34Y^{92} \\ & + 27Y^{93} + 24Y^{94} + 19Y^{95} + 16Y^{96} + 13Y^{97} + 10Y^{98} + 8Y^{99} + 7Y^{100} \\ & + 4Y^{101} + 4Y^{102} + 2Y^{103} + 3Y^{104} + 2Y^{106} + Y^{108}. \end{aligned}$$

$\zeta^{\triangleleft}_{\mathfrak{tr}_6(\mathbb{Z})}(s)$ is uniform.

## 3 Functional equation

The local zeta function satisfies the functional equation

$$\zeta^{\triangleleft}_{\mathfrak{tr}_6(\mathbb{Z}),p}(s)\Big|_{p \to p^{-1}} = -p^{15-86s}\zeta^{\triangleleft}_{\mathfrak{tr}_6(\mathbb{Z}),p}(s).$$

## 4 Abscissa of convergence and order of pole

The abscissa of convergence of $\zeta^{\triangleleft}_{\mathfrak{tr}_6(\mathbb{Z})}(s)$ is 6, with a simple pole at $s = 6$.

## 5 Ghost zeta function

This zeta function is its own ghost.

## 6 Natural boundary

$\zeta^{\triangleleft}_{\mathfrak{tr}_6(\mathbb{Z})}(s)$ has natural boundary at $\Re(s) = 0$.